Vibrational Analysis of an Isotropic Plate having Dual Surface Cracks with Two Boundary Conditions

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Abstract*— Rupturing of a plate under loading is a because of a sequence of processes which include progression and interaction of multiple micro-cracks. In this study an isotropic plate with multiple part through surface cracks in plate, consisting of continuous lines is considered. Using Kirchoff-love plate theory equation of motion is derived for a given set of boundary conditions leading to the nonlinear vibrations. The effects of through-thickness shear and rotary inertia are ignored for the sake of simplicity of vibrational problem, Berger's formulation is used to generate the form for the in-plane forces and make the model differential equation nonlinear. The developed non-linear equation is transformed into a time dependent modal co-ordinate by the application of Galerkin's method. Superposition and asymptotic approximation technique is used to evaluate the stresses generated in the close vicinity of cracks. Results are presented in terms of the first mode of natural frequency against the crack length and location in the plate. This method is efficient and easier to use.

***Keywords*— Vibration, Crack, Equilibrium, Nonlinear, Isotropic Plate, Galerkin's Method**

I. INTRODUCTION

In early 19th century, investigation on the effects of small cracks, their growth and their interactions in the field of aerospace as well as other naval and civil industries started. In addition to predicting the processes that lead to failure of structure, such investigations proved themselves to be fruitful in the development of high quality materials having improved resistance to fatigue.

The use of plate components is extensive in complex structural assemblies in aerospace, naval and civil industries [1-4]. Structural and mechanical gears require constant evaluation to avoid failure and to maintain their structural integrity. Flaws in a structure can arise from continuous loadings, corrosion, environmental factors such as temperature, dampness, rain and other external factors such as modifications in the general properties of the structure. Dynamic characteristics such as modal shapes, modal frequencies and modal damping values linked to each modal frequency can be varied by the existence of even a small crack. It is due to all these reasons that a need to understand the dynamics of a cracked structure is essential.

In order to reduce the odds of structural failure it is desirable to detect cracks when they are small. Now-a-days one can easily detect cracks using non-destructive techniques to detect cracks acoustic emission, X-rays, Ultrasonic testing and Vibrational techniques. Vibrational technique is famed for its time and cost saving. [5-9]. The dynamic characteristics of a plates with various shapes having no crack in them has been given in a study by Leissa in 1969 [10]. For surface cracks static solutions have been developed [4] Although mostly were done numerically. Rice and Levy in 1972 [11] developed a Line Spring Model (LSM) which proved itself to be productive in approximate analytical solution technique. The advantage of this model is the reduction of three dimensional problem into a two dimensional problem. Later, it was found that errors in this model can be reduced if the crack length is kept larger as compared to the crack thickness. Israr [1-3, 12] investigated the behavior of a thin plate consisting of a part-through crack located at the middle of the plate under the application of an external loading at a specific position with different sets of boundary conditions using the LSM, deducing that crack in the center of the plate has an effect on the natural frequency of the plate. Israr's work was further carried forward by the incorporation of orientation angle of the surface crack in a thin plate by Ismail [13] and Mohanty [4]. It was concluded that the natural frequency of the plate depends greatly on the crack length and its orientation angle.

On the other hand, very little research has been done in terms of obtaining an analytical model of multiple cracks having different geometrical configurations. Most research done in this domain was by Yijun Du and Atilla Aydin [38]. They used echelon array method to develop a model for multiple cracks. This research focuses on expanding and enhancing the previous work of Israr [1] by the integration of multiple cracks factor with different geometrical configurations in the plate. An approximate analytical model is derived of a rectangular plate having a part through crack at a random location with no orientation.

II. MATHEMATICAL MODELING

A. Rectangular Plate having a Part-Through Surface Crack at its Center

According to Kirchoff's Classical Plate Theory, consider a horizontal surface crack initially at the center of the plate with the forces and moments acting at the edge. Thus, by applying the moment and force equilibrium equations, our governing differential equation comes out to be:

$$D\left[\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right]=P_z-\rho h\frac{\partial^2 w}{\partial t^2}+\frac{\partial^2 \overline{M_y}}{\partial y^2} \quad (1)$$

Where D is *Flexural Rigidity;* $D=Eh^3/12(1-\nu^2)$, ν is the Poisson's Ratio of the material, ρ is the density of the material, h is the uniform thickness of the plate, E is the Modulus of Elasticity of the material, , w is the transverse deflection, P_z is the load per unit area which acts perpendicular on the surface of the plate at a random location. The over-bar ($\bar{}$) term $\overline{M_y}$ represent moment generated due to the presence of the crack.

Now, adding the membrane or the in-plane forces in the z-direction in (1), we get the following equation of motion for the cracked plate:

$$D\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)=P_z-\rho h\frac{\partial^2 w}{\partial t^2}+q_x\frac{\partial^2 w}{\partial x^2}+\overline{q_y}\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 \overline{M_y}}{\partial y^2} \quad (2)$$

The nomenclature for the in-plane forces per unit length is given by q_x, $\overline{q_y}$, where the over bar ($\bar{}$) denotes the membrane forces generated due to the crack per unit length.

In case of free vibrations, $P_z=0$ whereas the value of w should be taken as such that it satisfies the boundary conditions.

B. Formulation of Crack Terms

The model devised by Rice and Levy [13] is based on Kirchoff's bending theory for thin plates and shells. They found a rough relationship between insignificant tensile and bending stresses at the location of the crack. After some re-arrangement, in the relations, it can be deduced that $m_{mn}=6\sigma_{mn}$. A representation of these stresses is given in Fig. A1.

$$\overline{\sigma_{mn}}=\frac{2a}{(6\alpha_{tb}^{\infty}+\alpha_{tt}^{\infty})(1-\upsilon^2)h+2a}\sigma_{yy} \quad (3)$$

$$\overline{m_{mn}}=\frac{2a}{3\left(\frac{\alpha_{bt}^{\infty}}{6}+\alpha_{bb}^{\infty}\right)(3+\upsilon)(1-\upsilon)h+2a}M_{yy} \quad (4)$$

In the above equations m,n are transitional variables required for algebraic interpretation and their values can be m,n = 1,2. The tensile and bending stresses generated near the crack location are given by $\overline{\sigma_{mn}}$ *and* $\overline{m_{mn}}$, Whereas the bending and tensile stresses at the far sides of the plate are given by m_{mn} *and* σ_{mn}, in the above equation (a) is the half length of the crack, and (h) is the thickness of the plate.
α_{tt}^{∞} and α_{bb}^{∞} are called non-dimensional stretching and bending compliances, whereas $\alpha_{tb}^{\infty}=\alpha_{bt}^{\infty}$ are defined as the non-dimensional bending-stretching compliances at the crack center. $\alpha_{bb}^{\infty}, \alpha_{tt}^{\infty}, \alpha_{bt}^{\infty}=\alpha_{tb}^{\infty}$ are used to match the stretching and bending resistance for symmetric Mode I loading and can be found in Israr [1] and Rice and Levy [11]

III. FORMULATING RELATIONSHIP BETWEEN TWO MODE-1 HORZIONTAL CRACKS

Sneddon and Lowengrub [37] gave the logical expressions for stress and displacement fields linked with isolated mode-I crack. The formulation depends upon the region one is interested in because stresses and displacements vary from region to region in a cracked plate. For our current study we have considered approximate tri-polar coordinate expressions for stress in the close region of crack tips. It will help us in locating one crack with respect to other. (Figure A2).

Geometrical configuration of two Mode-1 cracks. Crack AB is located at the centre of the plate whereas the crack A'B' is located at an arbitrary position. In order to locate A'B' with respect to crack AB we have assumed the following geometry. Distance between two inner crack tips is given by S_1. Whereas, The angle between crack AB and A'B' is given by $\emptyset_1$. The distance between point O and the crack tip B is given by S. ε is the distance between the crack tip A' and point O. In the above case we have considered narrow spacing i.e. $S_1 << a$; where **a** is the half crack length. According to the formulae of linear fracture mechanics, because of Mode-1 loading the stresses generated near the crack tip are given by:

$$\sigma_{yy}=\sigma_y^{\infty}\sqrt{\frac{a}{\sqrt{2\varepsilon}}}+\sigma_y^{\infty}\sqrt{\frac{a}{\sqrt{2S}}}\left(\cos\emptyset_2\frac{1}{2}+\sin\emptyset_2\frac{1}{2}\sin\emptyset_2\frac{3}{2}\right)+\sigma_y^{\infty} \quad (5)$$

$$\sigma_{xy}=\sigma_y^{\infty}\sqrt{\frac{a}{\sqrt{2S}}}\left(\sin\emptyset_2\frac{1}{2}\cos\emptyset_2\frac{3}{2}\right) \quad (6)$$

The radial angle measured from crack AB plane is $\emptyset_2$ and is in counter-clockwise direction, σ_{yy} is the remote stress. Normal stress associated with the crack AB is the first term of the right-hand side equation (5). The contribution from shear stress for crack AB is zero. Stress contribution because of crack A'B' is given by the second term in equation (5) and the shear stresses contributed by the crack A'B' is given in equation (6). ε**,** the stress singularity parameter, It is worth noting that value of ε depends on the crack length as well as the distance between the two crack tips. Generally its value is very small. The normal stress near the tip of the crack can be formulated with the help of Kirchoff-love plate theory. According to our equation of crack formulation in equation (3)

$$\overline{\sigma_{mn}} = \frac{2a}{(6\alpha_{tb}^{\propto} + \alpha_{tt}^{\propto})(1-\upsilon^2)h + 2a}\sigma_{yy}$$

Putting value of remote stress σ_{yy} in above equation we have:

$$\overline{\sigma_{mn}} = \frac{2a}{(6\alpha_{tb}^{\propto} + \alpha_{tt}^{\propto})(1-\upsilon^2)h + 2a}\left[\begin{array}{c}\sigma_y^{\infty}\sqrt{\frac{a}{\sqrt{2\varepsilon}}} + \\ \sigma_y^{\infty}\sqrt{\frac{a}{\sqrt{2S}}}\left(cos\emptyset_2\frac{1}{2} + sin\emptyset_2\frac{1}{2}sin\emptyset_2\frac{3}{2}\right) + \\ \sigma_y^{\infty}\end{array}\right] \tag{7}$$

Ignoring the effect of shear stresses, because for Mode-1 cracks the values of shear stresses are very small. In linear elastic fracture mechanics; for two cracks present in close proximity i.e. $S_1 << a$; the value of $\emptyset_3 = 0$ and $\varepsilon/a << 1$. For narrow spacing the value of $\emptyset_2$ can be evaluated in terms of angle $\emptyset_1$ which is specified in (Figure A3).

$$sin\emptyset_2 = \frac{S_1 sin\emptyset_1}{\sqrt{[(S_1 sin\emptyset_1)^2 + (S_1 cos\emptyset_1 - \varepsilon)^2]}} \tag{8}$$

$$\emptyset_2 = sin^{-1}\left[\frac{S_1 sin\emptyset_1}{\sqrt{[(S_1 sin\emptyset_1)^2 + (S_1 cos\emptyset_1 - \varepsilon)^2]}}\right] \tag{9}$$

Where S_1 and $\emptyset_1$ are defined as the distance and angle between the two crack tips. In order to calculate the value of ε we can use multiple ways. One is from the analytical solutions given in the literature and the other method is from the empirical expressions. But the necessity of calculating value of ε is only valued for wide spacing.

The bending and tensile stresses on the extreme ends of the plate were developed by [5, 14] after making use of the relationships provided by [21]. These relationships are given as follows:

$$\left.\begin{array}{l}\sigma_{mn} = \frac{q_{mn}}{h} = \frac{1}{h}\int_{-\frac{h}{2}}^{+\frac{h}{2}}\tau_{mn}(x,y,z)dz \\ m_{mn} = \frac{6}{h^2}M_{mn} = \frac{6}{h^2}\int_{-\frac{h}{2}}^{+\frac{h}{2}}z\tau_{mn}(x,y,z)dz\end{array}\right\} \tag{10}$$

Hence, we have:

$$\overline{q_y} = -\overline{q_{mn}} = \frac{2a}{(6\alpha_{tb}^{\propto} + \alpha_{tt}^{\propto})(1-\upsilon^2)h + 2a}\left[\begin{array}{c}\sigma_y^{\infty}\sqrt{\frac{a}{\sqrt{2\varepsilon}}} + \\ \sigma_y^{\infty}\sqrt{\frac{a}{\sqrt{2S}}}\left(cos\emptyset_2\frac{1}{2} + sin\emptyset_2\frac{1}{2}sin\emptyset_2\frac{3}{2}\right) \\ +\sigma_y^{\infty}\end{array}\right] \tag{12}$$

Also,

$$\overline{M_y} = \overline{m_{mn}} = \frac{2a}{3\left(\frac{\alpha_{bt}^{\propto}}{6} + \alpha_{bb}^{\propto}\right)(3+\upsilon)(1-\upsilon)h + 2a}M_{yy} \tag{13}$$

Thus, we can finally determine the relationship between nominal tensile stress $\overline{\sigma_{mn}}$, nominal bending stress $\overline{m_{mn}}$ near the crack location with σ_{yy}&M_{yy}, the tensile stresses and bending moments at the extreme end of plate as:

$$\begin{aligned} &D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) \\ &= P_z - \rho h\frac{\partial^2 w}{\partial t^2} + q_x\frac{\partial^2 w}{\partial x^2} \\ &- \left[\frac{2a}{(6\alpha_{tb}^{\propto} + \alpha_{tt}^{\propto})(1-\upsilon^2)h + 2a}\left[\begin{array}{c}\sigma_y^{\infty}\sqrt{\frac{a}{\sqrt{2\varepsilon}}} + \\ \sigma_y^{\infty}\sqrt{\frac{a}{\sqrt{2S}}}\left(cos\emptyset_2\frac{1}{2} + sin\emptyset_2\frac{1}{2}sin\emptyset_2\frac{3}{2}\right) + \\ \sigma_y^{\infty}\end{array}\right]\right]\frac{\partial^2 w}{\partial y^2} \\ &+ \frac{\partial^2}{\partial y^2}\left[\frac{2a}{3\left(\frac{\alpha_{bt}^{\propto}}{6} + \alpha_{bb}^{\propto}\right)(3+\upsilon)(1-\upsilon)h + 2a}M_{yy}\right]\end{aligned} \tag{14}$$

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IV. COMPARISON WITH RESULTS

Our study on the effect of the presence of dual cracks on the natural frequency of the plate with two boundary conditions (SSSS and CCSS) was done on Aluminum alloy 5083. The results show that the reduction in overall first mode natural frequency of the plate is due presence of a single crack, a statement that can be justified from the fact that the presence of small discontinuity reduces the stiffness of the plate. The natural frequency is also affected by the geometrical parameters of the plate, i.e. its length and thickness. Also, increment in crack length can cause a significant reduction in the amount of natural frequency of the plate. The author's formulation can be compared with existing work of Israr [1] and it can be seen from (Figure A3) that for only one horizontal crack at the center of the plate; both the graphs overlap one another, indicating that the error between the results is insignificant.

(Figure A4) shows that there is significant reduction in the stiffness of the plate due to presence of an additional crack, hence, reduction in the natural frequency of the plate is also imminent. This effect is seen to be higher if both the cracks are very close to each other. As the distance between them increases, the natural frequency increases slightly.

A. *Figures and Tables*

Figure A1 (Part though surface crack at the center of the plate)

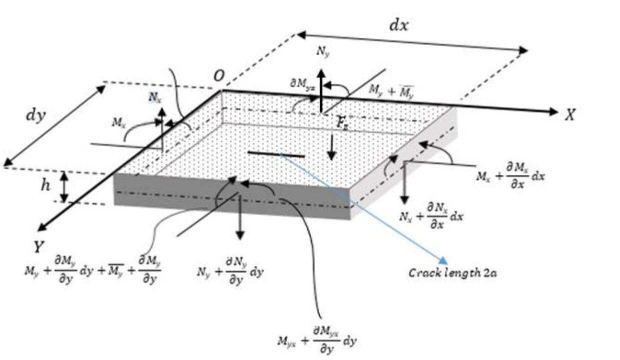


Figure A2 (2 Part though surface cracks one at the center and other randomly located in the plate)

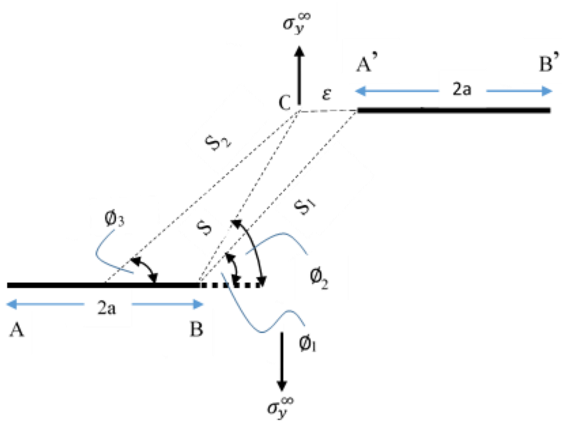
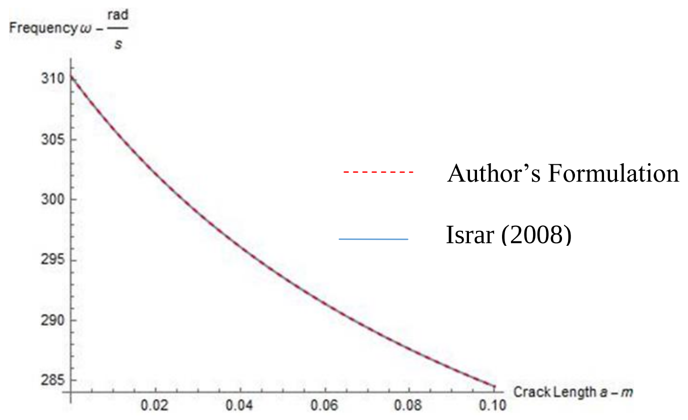


Figure A3 (Comparison of the Effect on the Natural Frequency of the Plate with increasing Crack Length for SSSS Boundary Condition)



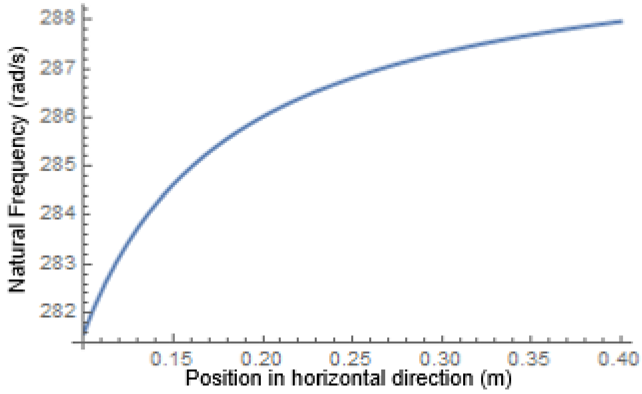


Figure A4 (Effect of the Distance between the two crack lengths for CCSS Boundary condition)

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REFERENCES

[1] A. Israr, Vibration analysis of cracked aluminium plates, in, University of Glasgow, 2008.

[2] A. Israr, M.P. Cartmell, E. Manoach, I. Trendafilova, M. Krawczuk, Ĺ. Arkadiusz, Analytical modeling and vibration analysis of partially cracked rectangular plates with different boundary conditions and loading, Journal of Applied Mechanics, 76 (2009) 011005.

[3] A. Israr, S. Zulfiqar, Nonlinear Vibration of Partially Cracked Plates Using Higher Order Perturbation Method, Journal of Space Technology, 1 (2012) 7-10.

[4] T. Bose, A. Mohanty, Vibration analysis of a rectangular thin isotropic plate with a part-through surface crack of arbitrary orientation and position, Journal of Sound and Vibration, 332 (2013) 7123-7141.

[5] S. Neogy, V. Ramamurti, Effect of Crack on the Natural Frequency of Rotating Twisted Blades, in: Proc. of DETC.

[6] V. Ramamurti, S. Neogy, Effect of Cracks on the Natural Frequency of Cantilevered Plates—A Rayleigh-Ritz Solution∗, Journal of Structural Mechanics, 26 (1998) 131-143.
[7] I. Trendafilova, A study on vibration-based damage detection and location in an aircraft wing scaled model, in: Applied Mechanics and Materials, Trans Tech Publ, 2005, pp. 309-314.
[8] I. Trendafilova, An investigation on vibration-based damage detection in an aircraft wing scaled model, Key Engineering Materials, 293 (2005) 321-328.
[9] I. Trendafilova, E. Manoach, M.P. Cartmell, M. Krawczuk, W.M. Ostachowicz, M. Palacz, On the problem for damage detection of vibrating cracked plates, Applied Mechanics and Materials, 5 (2006) 247-254.
[10] A.W. Leissa, Vibration of plates, in, DTIC Document, 1969.
[11] J. Rice, N. Levy, The part-through surface crack in an elastic plate, Journal of Applied Mechanics, 39 (1972) 185-194.
[12] A. Israr, L. Atepor, Investigation of the nonlinear dynamics of a partially cracked plate, in: Journal of Physics: Conference Series, IOP Publishing, 2009, pp. 012087.
[13] R. Ismail, Vibration analysis of a plate with an arbitrarily orientated surface crack, in, University of Glasgow, 2013.
[14] S. Khadem, M. Rezaee, An analytical approach for obtaining the location and depth of an all-over part-through crack on externally in-plane loaded rectangular plate using vibration analysis, Journal of sound and vibration, 230 (2000) 291-308.
[15] S. Khadem, M. Rezaee, Introduction of modified comparison functions for vibration analysis of a rectangular cracked plate, Journal of Sound and Vibration, 236 (2000) 245-258.
[16] S. Yoo, H. Kwak, B. Kim, Detection and location of a crack in a plate using modal analysis, in: Proceedings of International Modal Analysis Conference (IMAC-XVII), Orlando, Florida, 1999, pp. 1902-1908.
[17] R. Ismail, M. Cartmell, An analysis of the effects of the orientation angle of a surface crack on the vibration of an isotropic plate, in: Journal of Physics: Conference Series, IOP Publishing, 2012, pp. 012007.
[18] S. Timoshenko, S. Woinowsky-Krieger, S. Woinowsky-Krieger, Theory of plates and shells, McGraw-hill New York, 1959.
[19] R. Szilard, Theory and analysis of plates, (1974).
[20] R. King, Elastic-plastic analysis of surface flaws using a simplified line-spring model, Engineering Fracture Mechanics, 18 (1983) 217-231.
[21] Z. Zhao-Jing, D. Shu-Ho, Stress intensity factors for an inclined surface crack under biaxial stress state, Engineering fracture mechanics, 47 (1994) 281-289.
[22] N.I. Muskhelishvili, Some basic problems of the mathematical theory of elasticity, Springer Science & Business Media, 1977.
[23] P. Theocaris, J. Michopoulos, A closed-form solution of a slant crack under biaxial loading, Engineering Fracture Mechanics, 17 (1983) 97-123.
[24] P. Joseph, F. Erdogan, Surface crack in a plate under antisymmetric loading conditions, International Journal of Solids and Structures, 27 (1991) 725-750.
[25] Y. Lu, Y. Xu, Line-spring model for a surface crack loaded antisymmetrically, National University of Defense Technology Technical Paper, (1986).
[26] H. Okamura, H. Liu, C.-S. Chu, H. Liebowitz, A cracked column under compression, Engineering fracture mechanics, 1 (1969) 547-564.
[27] C. Vendhan, A study of Berger equations applied to non-linear vibrations of elastic plates, International Journal of Mechanical Sciences, 17 (1975) 461-468.
[28] J.-M. Berthelot, Dynamics of composite materials and structures, Institute for Advanced Materials and Mechanics, Springer, New York, (1999).
[29] Z. Fan, Transient vibration and sound radiation of a rectangular plate with viscoelastic boundary supports, International Journal for Numerical Methods in Engineering, 51 (2001) 619-630.
[30] T. Wah, The normal modes of vibration of certain nonlinear continuous systems, Journal of Applied Mechanics, 31 (1964) 139-140.
[31] J. Ramachandran, D. Reddy, Nonlinear vibrations of rectangular plates with cutouts, AIAA Journal, 10 (1972) 1709-1710.
[32] B. Stahl, L. Keer, Vibration and stability of cracked rectangular plates, International Journal of Solids and Structures, 8 (1972) 69-91.
[33] R. Solecki, Bending vibration of a simply supported rectangular plate with a crack parallel to one edge, Engineering Fracture Mechanics, 18 (1983) 1111-1118.
[34] Q. Guan-Liang, G. Song-Nian, J. Jie-Sheng, A finite element model of cracked plates and application to vibration problems, Computers & structures, 39 (1991) 483-487.
[35] M. Krawczuk, A rectangular plate finite element with an open crack, Computers & structures, 46 (1993) 487-493.
[36] M. Krawczuk, A. Żak, W. Ostachowicz, Finite element model of plate with elasto-plastic through crack, Computers & Structures, 79 (2001) 519-532.

[37] Pompe, W. "IN Sneddon/M. Lowengrub, Crack Problems in the Classical Theory of Elasticity.(The SIAM Series in Applied Mathematics). VIII+ 221 S. m. 57 Fig. New York/London/Sydney/Toronto 1969. John Wiley & Sons, Inc. Preis geb. 140 s. net." ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik 51.3 (1971): 238-239.
[38] Du, Yijun, and Atilla Aydin. "Interaction of multiple cracks and formation of echelon crack arrays." International Journal for Numerical and Analytical Methods in Geomechanics 15.3 (1991): 205-218.