Linear Programming Problems

- Two common formulations of linear programming (LP) problems are:

$$\min_{x_j} \ z = \sum_{j=1}^{n} c_j x_j$$

Subject to: $\sum_{j=1}^{n} a_{ij} x_j \geq b_i\,, \quad i = 1,2, \dots, m; \quad x_j^L \leq x_j \leq x_j^U, j = 1, \dots, n$

$$\max_{x_j} \ z = \sum_{j=1}^{n} c_j x_j$$

Subject to: $\sum_{j=1}^{n} a_{ij} x_j \leq b_i\,, \quad i = 1,2, \dots, m; \quad x_j^L \leq x_j \leq x_j^U, j = 1, \dots, n$

Linear Programming Problems

- The standard LP problem is given as:

$$\min_{x_i} \; z = \sum_{j=1}^{n} c_j x_j$$

Subject to: $\sum_{j=1}^{n} a_{ij} x_j = b_i$, $x_j \geq 0$; $i = 1,2, \ldots, m$

- Or, in matrix form as:

$\min_{\boldsymbol{x}} z = \boldsymbol{c}^T \boldsymbol{x}$, subject to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, $\boldsymbol{x} \geq \boldsymbol{0}$

where $\boldsymbol{A} \in \mathbb{R}^{m \times n}$; $\boldsymbol{x}, \boldsymbol{c} \in \mathbb{R}^n$, $\boldsymbol{b} \in \mathbb{R}^m$; $rank(\boldsymbol{A}) = m$

- The standard LP problem has the following characteristics:
  - It involves minimization of a scalar cost function.
  - The variables can only take on non-negative values, i.e., $x_j \geq 0$.
  - The right hand side (rhs) is assumed to be non-negative, i.e., $b_i \geq 0$.
  - The constraints are assumed to be linearly independent, $rank(\boldsymbol{A}) = m$.
  - The problem is assumed to be well-formulated, i.e., $\min_{\boldsymbol{x}} \boldsymbol{c}^T \boldsymbol{x} < \infty$.

Formulation of the Standard LP Problem

- When encountered, exceptions to the standard LP problem formulation are dealt as follows:
  - A maximization problem is changed to a minimization problem by taking negative of the cost function, i.e.,

    $$\max_{\boldsymbol{x}} \boldsymbol{c}^T \boldsymbol{x} \equiv \min_{\boldsymbol{x}} (-\boldsymbol{c}^T \boldsymbol{x}).$$
  - Any constant terms in z can be dropped.
  - Any $x_i \in \mathbb{R}$ (unrestricted in sign) are replaced by $x_i = x_i^+ - x_i^-$ where $x_i^+, x_i^- \geq 0$.
  - The inequality constraints are converted to equality constraints by the addition of slack variables (to LE constraint) or subtraction of surplus variables (from GE constraint).
  - If any $b_i < 0$, the constraint is first multiplied by -1, followed by the introduction of slack or surplus variables.

Solution to the LP Problem

- The LP problem is convex as the objective function and the constraints are linear; therefor a global optimum exists
- The feasible region for the LP problem is defined by:
  $\mathcal{S} = \{\boldsymbol{x}: \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$

  The feasible region is a polytope (polygon in n dimensions)
- Due to only equality constraints, the optimal solution lies at the boundary of feasible region (one or more constraints must be active at the optimum)

Basic LP Solutions

- A basic solution $\boldsymbol{x}$ to the LP problem satisfies two conditions:
  - $\boldsymbol{x}$ is a solution to $\boldsymbol{Ax} = \boldsymbol{b}$
  - The columns of $\boldsymbol{A}$ corresponding to the nonzero components of $\boldsymbol{x}$ are linearly independent
- Since rank $(\boldsymbol{A}) = m$, $\boldsymbol{x}$ has at the most m nonzero components. Thus, a basic solution is obtained by choosing $n - m$ variables as 0
- In terms of the canonical form, $\boldsymbol{I}_{(m)}\boldsymbol{x}_{(m)} + \boldsymbol{Q}\boldsymbol{x}_{(n-m)} = \boldsymbol{b}'$, a basic solution is given as: $\boldsymbol{x}_{(m)} = \boldsymbol{b}'$
- The total number of basic solutions is finite, and is given as: $\binom{n}{m} = \frac{n!}{m!(n-m)!}$
- A basic solution that is also feasible is a basic feasible solution (BFS)

The Simplex Method

- The simplex method iteratively solves the LP problem
- The steps in the Simplex methods are:
  1. Initialize from a known BFS, with m non-zero values
  2. Check optimality
  3. Replace a variable in the basis with a previously nonbasic variable, such that the objective function value decreases
  4. Go to step 2
- An optimum is reached when no neighboring BFS with a lower objective function value can be found

The Simplex Method

- At any step we can partition the problem into basic and nonbasic variables as: $\boldsymbol{x}^T = [\boldsymbol{x}_B, \boldsymbol{x}_N], \boldsymbol{c}^T = [\boldsymbol{c}_B, \boldsymbol{c}_N], \boldsymbol{A} = [\boldsymbol{B},\ \boldsymbol{N}],$ to write

$$\min_{\boldsymbol{x}} z = \boldsymbol{c}_B^T \boldsymbol{x}_B + \boldsymbol{c}_N^T \boldsymbol{x}_N,$$

Subject to $\boldsymbol{B}\boldsymbol{x}_B + \boldsymbol{N}\boldsymbol{x}_N = \boldsymbol{b},\ \boldsymbol{x}_B \geq \boldsymbol{0},\ \boldsymbol{x}_N \geq \boldsymbol{0}$

- The current basic solution is given as: $\boldsymbol{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b},\ \boldsymbol{x}_N = \boldsymbol{0}$.
- The matrix $\boldsymbol{B}$ is called the basis matrix and its columns are the basis vectors.

The Simplex Method

- Consider the constraint: $\boldsymbol{B}\boldsymbol{x}_B + \boldsymbol{N}\boldsymbol{x}_N = \boldsymbol{b}$, and assume that we want to study the change in the cost function when $x_j \in \boldsymbol{x_N}$ is assigned a nonzero value
- Accordingly, use $\boldsymbol{x}_B = \boldsymbol{B}^{-1}(\boldsymbol{b} - \boldsymbol{N}\boldsymbol{x}_N)$ to express the cost as:

  $z = \boldsymbol{c}_B^T\boldsymbol{x}_B + \boldsymbol{c}_N^T\boldsymbol{x}_N = \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{b} + (\boldsymbol{c}_N^T - \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{N})\boldsymbol{x}_N$

  or, $\quad z = \boldsymbol{y}^T\boldsymbol{b} + \hat{\boldsymbol{c}}_N^T\boldsymbol{x}_N = \hat{z} + \hat{\boldsymbol{c}}_N^T\boldsymbol{x}_N$
- In the above
  - $\boldsymbol{y}^T = \boldsymbol{c}_B^T\boldsymbol{B}^{-1}$ is a vector of Simplex (Lagrange) multipliers; any $y_i > 0$ represents an active constraint
  - $\hat{z} = \boldsymbol{y}^T\boldsymbol{b}$ represents the current optimal
  - $\hat{\boldsymbol{c}}_N^T = \boldsymbol{c}_N^T - \boldsymbol{y}^T\boldsymbol{N}$ represent the reduced costs for nonbasic variables; any $\hat{c}_j < 0$ represents the potential to reduce $\hat{z}$

Entering and Leaving Basic Variables

- A previously nonbasic variable with $\hat{c}_q < 0$ that is assigned a nonzero value δ_q in order to reduce z is termed an entering basic variable (EBV).
- The update to the current basic solution $\boldsymbol{x}_B$ by the introduction of EBV is given as: $\boldsymbol{x}_B = \boldsymbol{B}^{-1}\left(\boldsymbol{b} - \boldsymbol{A}_q x_q\right) = \widehat{\boldsymbol{b}} - \widehat{\boldsymbol{A}}_q x_q$, where $\boldsymbol{A}_q$ represents the qth column of $\boldsymbol{A}$ that corresponds to the EBV.
- In order to maintain feasibility ($\boldsymbol{x}_B \geq \boldsymbol{0}$), the maximum allowable value of x_q is given as: $\delta_q = \frac{\hat{b}_p}{\hat{A}_{p,q}} = \min_i \left\{ \frac{\hat{b}_i}{\hat{A}_{i,q}} : \hat{A}_{i,q} > 0 \right\}$. The element $\hat{A}_{p,q}$ is termed as the pivot element.
- Assigning maximum value to x_q results in one of the previously basic variable taking on zero value; that variable is termed as leaving basic variable (LBV).

The Simplex Algorithm

- Initialize: find an initial BFS to start the algorithm; accordingly, determine $\boldsymbol{x_B},\ \boldsymbol{x_N},\ \boldsymbol{B},\ \boldsymbol{N},\ \boldsymbol{y}^T = \boldsymbol{c}_B^T\boldsymbol{B}^{-1},\ \ \hat{z} = \ \boldsymbol{y}^T\boldsymbol{b}$.
- Optimality test: compute $\hat{\boldsymbol{c}}_N^T = \boldsymbol{c}_N^T - \boldsymbol{y}^T\boldsymbol{N}$. If all $\hat{c}_j > 0$, the optimal has been reached; otherwise, select a variable x_q with $\hat{c}_q < 0$ as EBV.
- Ratio test: compute $\widehat{\boldsymbol{A}}_q = \boldsymbol{B}^{-1}\boldsymbol{A}_q$ associated with EBV, and determine: $\delta_q = \frac{\hat{b}_p}{\hat{A}_{p,q}} = \min_i \left\{\frac{\hat{b}_i}{\hat{A}_{i,q}} : \hat{A}_{i,q} > 0\right\}$.
- Update: assign $x_q \leftarrow \delta_q,\ \boldsymbol{x}_B \leftarrow \widehat{\boldsymbol{b}} - \hat{A}_q x_q,\ \hat{z} \leftarrow \hat{z} + \hat{c}_q x_q$; update $\boldsymbol{x_B},\ \boldsymbol{x_N},\ \boldsymbol{B},\ \boldsymbol{N}$.

Tableau Implementation of the Simplex Algorithm

- A tableau is an augmented matrix that represents the current BFS in the form: $\boldsymbol{I}_m \boldsymbol{x}_B + \boldsymbol{B}^{-1} \boldsymbol{N} \boldsymbol{x}_N = \boldsymbol{b}'$
- Each row of the tableau represents a constraint equation and each column represents a variable, with the columns associated with basic variables identified by unit vectors.
- Additionally, the last row, termed as the cost function row, includes the reduced costs and the current optimum.
- When an EBV and a pivot element $\hat{A}_{p,q}$ has been identified, Gauss-Jordan eliminations are used to reduce A_q to a unit vector.
- The tableau method implements the simplex algorithm by iteratively computing the inverse of the basis ($\boldsymbol{B}$) matrix.

Example

- Consider the LP problem:

$$\max_{x_1,x_2} z = 3x_1 + 2x_2$$

Subject to: $2x_1 + x_2 \leq 12,\ \ 2x_1 + 3x_2 \leq 16;\ x_1 \geq 0, x_2 \geq 0$

- Convert the problem to standard LP form as:

$$\min_{x_1,x_2} z = -3x_1 - 2x_2$$

Subject to: $2x_1 + x_2 + s_1 = 12,\ \ 2x_1 + 3x_2 + s_2 = 16;\ x_1 \geq 0, x_2 \geq 0, s_1 \geq 0, s_2 \geq 0$

- Construct the initial Tableau:

Basic	x_1	x_2	s_1	s_2	Rhs
s_1	2	1	1	0	12
s_2	2	3	0	1	16
$-z$	-3	-2	0	0	0

EBV: x_1, LBV: s_1, pivot: (1,1)

Example: Simplex Iterations

Basic	x_1	x_2	s_1	s_2	Rhs
s_1	2	1	1	0	12
s_2	2	3	0	1	16
$-z$	-3	-2	0	0	0

EBV: x_1, LBV: s_1, pivot: (1,1)

Basic	x_1	x_2	s_1	s_2	Rhs
x_1	1	0.5	0.5	0	6
s_2	0	2	-1	1	4
$-z$	0	-0.5	1.5	0	18

EBV: x_2, LBV: s_2, pivot: (2,2)

Basic	x_1	x_2	s_1	s_2	Rhs
x_1	1	0	0.75	-0.25	5
x_2	0	1	-0.5	0.5	2
$-z$	0	0	1.25	0.25	19

Optimal solution: $x_1^* = 5, x_2^* = 2, z_{opt} = -19$

Two Phase Simplex Method

- An initial BFS is not obvious for problems formulated with EQ and GE type constraints (origin not included in the feasible region)
- Add auxiliary variables $\tilde{\boldsymbol{x}}$ to all EQ and GE type constraints
- Define an auxiliary problem to be solved Phase I:

  $$\min_{\tilde{x}_i} \sum_{i=1}^{m} \tilde{x}_i$$

  Subject to: $\boldsymbol{A}\boldsymbol{x} + \tilde{\boldsymbol{x}} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}, \tilde{\boldsymbol{x}} \geq \boldsymbol{0}$
- An initial BFS for the auxiliary problem is given as: $\tilde{\boldsymbol{x}} = \boldsymbol{b}$
- Add an auxiliary objective row to the simplex tableau
- Bring auxiliary variables into the basis
- Proceed with simplex iterations using auxiliary cost function row
- As the optimum value for the auxiliary objective is zero, Phase I ends when auxiliary objective becomes zero

Example: Two Phase Simplex

- Consider the following LP problem:

$$\max_{x_1,x_2} z = 3x_1 + 2x_2$$

Subject to: $3x_1 + 2x_2 \geq 12,\ 2x_1 + 3x_2 \leq 16, x_1 \geq 0, x_2 \geq 0$

- Convert to standard LP form:

$$\min_{x_1,x_2} z = -3x_1 - 2x_2$$

Subject to: $3x_1 + 2x_2 - \mathrm{s}_1 = 12,\ 2x_1 + 3x_2 + \mathrm{s}_2 = 16;$
$x_1, x_2, s_1, s_2 \geq 0$

- Define the auxiliary problem:

$$\min_{x_1,x_2} z_a = a_1$$

Subject to: $3x_1 + 2x_2 - s_1 + a_1 = 12,\ 2x_1 + 3x_2 + s_2 = 16;$
$x_1, x_2, s_1, s_2, a_1 \geq 0$

Example: Phase I

Basic	x_1	x_2	s_1	s_2	a_1	Rhs
	3	2	-1	0	1	12
s_2	2	3	0	1	0	16
$-z$	-3	-2	0	0	0	0
$-z_a$	0	0	0	0	1	0

Basic	x_1	x_2	s_1	s_2	a_1	Rhs
s_1	3	2	-1	0	1	12
s_2	2	3	0	1	0	16
$-z$	-3	-2	0	0	0	0
$-z_a$	-3	-2	1	0	0	-12

EBV: x_1, LBV: s_1, pivot: (1,1)

Basic	x_1	x_2	s_1	s_2	a_1	Rhs
x_1	1	2/3	-1/3	0	1/3	4
s_2	0	5/3	2/3	1	-2/3	8
$-z$	0	0	-1	0	1	12
$-z_a$	0	0	0	0	1	0

Example: Phase II

Basic	x_1	x_2	s_1	s_2	Rhs
x_1	1	2/3	-1/3	0	4
s_2	0	5/3	2/3	1	8
$-z$	0	0	-1	0	12

EBV: s_1, LBV: s_2, pivot: (2,3)

Basic	x_1	x_2	s_1	s_2	Rhs
x_1	1	3/2	0	1/2	8
s_1	0	5/2	1	3/2	12
$-z$	0	5/2	0	3/2	24

$x_1^* = 8, \quad x_2^* = 0, \quad z^* = -24$

Simplex Algorithm: Abnormal Terminations

- If the reduced cost y_j for a nonbasic variable in the final tableau is zero, then there are possibly multiple optimum solutions with equal cost function value. This happens when cost function contours (level curves) are parallel to one of the constraint boundaries.
- If the reduced cost is negative but the pivot step cannot be completed due to all coefficients in the LBV column being negative, it reveals a situation where the cost function is unbounded below.
- If, at some point during Simplex iterations, a basic variable attains a zero value, it is called degenerate variable and the current BFS is termed as degenerate solution. The degenerate row hence forth becomes the pivot row with no further improvement in the objective function.

Post Optimality Analysis

- Post optimality analysis explores the effects of parametric changes on the optimal solution.
- There are five basic parametric changes affecting the LP solution:
  - Changes in cost function coefficients, c_j, which affect the level curves of the cost function.
  - Changes in resource limitations, b_i, which affect the set of active constraints.
  - Changes in constraint coefficients, a_{ij}, which affect the active constraint gradients.
  - The effect of including additional constraints
  - The effect of including additional variables

Postoptimality Analysis

- The current cost function value in the Simplex algorithm is given as:
  $z = \boldsymbol{y}^T \boldsymbol{b} + \hat{\boldsymbol{c}}_N^T \boldsymbol{x}_N$, where $\boldsymbol{y}^T = \boldsymbol{c}_B^T \boldsymbol{B}^{-1}$ and $\hat{\boldsymbol{c}}_N^T = \boldsymbol{c}_N^T - \boldsymbol{y}^T \boldsymbol{N}$
- The cost function is expanded as: $z = \sum_i y_i b_i + \sum_j \hat{c}_j x_j$,
  where $\hat{c}_j = c_j - \boldsymbol{y}^T \delta \boldsymbol{N}_j$, and $\boldsymbol{N}_j$ represents the jth column of $\boldsymbol{N}$.
- Then, taking the differentials with respect to b_i, c_j, we obtain:
  $\delta z = \sum_i y_i \delta b_i + \sum_j \delta \hat{c}_j x_j$, $\delta \hat{c}_j = \delta c_j - \boldsymbol{y}^T \delta \boldsymbol{N}_j$
- The above formulation may be used to analyze the effects of changes to b_i, c_j, and $\boldsymbol{N}_j$ on z.

Recovery of the Lagrange Multipliers

- The Lagrange multipliers can be recovered from the final tableau as follows:
  - For LE constraint, the Lagrange multiplier, $y_j \geq 0$, equals the reduced cost coefficient in the slack variable column.
  - For GE/EQ constraint, the Lagrange multiplier equals the reduced cost coefficient in the artificial variable column; where $y_j \leq 0$ for GE type, and y_j is unrestricted in sign for EQ type constraint.

Final Tableau Properties

- Consider the standard LP formulation of a maximization problem:

  $$\min_{\boldsymbol{x}} \; z = -\boldsymbol{c}^T\boldsymbol{x}$$

  Subject to: $\boldsymbol{Ax} + \boldsymbol{Is} = \boldsymbol{b},\;\; \boldsymbol{x} \geq \boldsymbol{0}$

- The initial and tableaus for the problem are given as:

Basic	$\boldsymbol{x}$	$\boldsymbol{s}$	Rhs
$\boldsymbol{s}$	$\boldsymbol{A}$	$\boldsymbol{I}$	$\boldsymbol{b}$
$-\boldsymbol{z}$	$-\boldsymbol{c}^T$	$\boldsymbol{0}$	0

Basic	$\boldsymbol{x}$	$\boldsymbol{s}$	Rhs
$\boldsymbol{x}_{\boldsymbol{B}}$	$\widetilde{\boldsymbol{A}}$	$\boldsymbol{S}$	$\widetilde{\boldsymbol{b}}$
$-\boldsymbol{z}$	$\hat{\boldsymbol{c}}^T$	$\boldsymbol{y}^T$	z^*

- These tableaus are related as: $[Tab]_{\text{final}} = \begin{bmatrix} \boldsymbol{S} & \boldsymbol{0} \\ \boldsymbol{y}^T & 1 \end{bmatrix} [Tab]_{\text{initial}}$

  Or $\;\widetilde{\boldsymbol{A}} = \boldsymbol{SA},\;\; \widetilde{\boldsymbol{b}} = \boldsymbol{Sb},\;\; \hat{\boldsymbol{c}}^T = \boldsymbol{y}^T\boldsymbol{A} - \boldsymbol{c}^T,\;\; z^* = \boldsymbol{y}^T\boldsymbol{b}$

- Thus, only $\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}^T, \boldsymbol{y}^T, \boldsymbol{S}$ need to be stored to recover the final tableau when the algorithm terminates

Postoptimality Analysis

- **Changes to the resource constraints (rhs).** A change in b_i has the effect of moving the associated constraint boundary. Then,
  - If the constraint is currently active ($y_i > 0$), the change will affect the current basic solution, $x_{\boldsymbol{B}} = \tilde{\boldsymbol{b}}$, as well as z_{opt}. If the new $x_{\boldsymbol{B}}$ is feasible, then $z_{opt} = \boldsymbol{y}^T \boldsymbol{b}$ is the new optimum value. If the new $x_{\boldsymbol{B}}$ is infeasible, then dual Simplex steps may be used to restore feasibility.
  - If the constraint is currently non-active ($y_i = 0$), then z_{opt} and $x_{\boldsymbol{B}}$ are not affected.

Postoptimality Analysis

- **Changes to the objective function coefficients.** Changes to c_j affect the level curves of z. Then,
  - If $c_j \in \boldsymbol{c}_B$, then since the new $\hat{c}_j \neq 0$, Gauss-Jordan eliminations are needed to return x_j to the basis. If optimality is lost in the process (any $\hat{c}_j < 0$), further Simplex steps will be needed to restore optimality. If optimality is not affected, then $z_{opt} = \boldsymbol{y}^T\boldsymbol{b}$ is the new optimum.
  - If $c_j \in \boldsymbol{c}_N$, though it does not affect z, still $\hat{c}_j$ needs to be recomputed and checked for optimality.

Postoptimality Analysis

- **Changes to the coefficient matrix.** Changes to the coefficient matrix affect the constraint boundaries. For a change in $\boldsymbol{A}_j$ (jth column of $\boldsymbol{A}$),
  - If $\boldsymbol{A}_j \in \boldsymbol{B}$, then Gauss-Jordan eliminations are needed to reduce A_j to a unit vector; then $\hat{c}_j$ needs to be recomputed and checked for optimality.
  - If $\boldsymbol{A}_j \in \boldsymbol{N}$, then the reduced cost $\hat{c}_j$ needs to be recomputed and checked for optimality.

Postoptimality Analysis

- **Adding Variables to the Problem.** If we add a new variable x_{n+1} to the problem, then
  - The cost function is updated as: $z = \boldsymbol{c}^T \boldsymbol{x} + c_{n+1} x_{n+1}$.
  - In addition, a new column A_{n+1} is added to the constraint matrix.
  - The associated reduced cost is computed as: $c_{n+1} - \boldsymbol{y}^T A_{n+1}$. Then, if this cost is positive, optimality is maintained; otherwise, further Simplex iterations are needed to recover optimality.

Postoptimality Analysis

- **Adding inequality Constraints.** Adding a constraint adds a row and the associated slack/surplus variable adds a column to the tableau. In this case, we need to check if adding a column to the basis affects the current optimum. Define an augmented $\boldsymbol{B}$ matrix as: $\boldsymbol{B} = \begin{bmatrix} \boldsymbol{B} & 0 \\ \boldsymbol{a}_B^T & 1 \end{bmatrix}$, where $\boldsymbol{B}^{-1} = \begin{bmatrix} \boldsymbol{B}^{-1} & 0 \\ \boldsymbol{a}_B^T \boldsymbol{B}^{-1} & 1 \end{bmatrix}$, and write the augmented final tableau as:

Basic	$\boldsymbol{x}_B$	$\boldsymbol{x}_N$	Rhs
$\boldsymbol{x}_B$	$\boldsymbol{I}$	$\boldsymbol{B}^{-1}\boldsymbol{N}$	$\boldsymbol{B}^{-1}\boldsymbol{b}$
$\boldsymbol{x}_{n+1}$	$\boldsymbol{I}$	$\boldsymbol{a}_B^T\boldsymbol{B}^{-1}\boldsymbol{N}$	$\boldsymbol{a}_B^T\boldsymbol{B}^{-1}\boldsymbol{b} + b_{n+1}$
$-\boldsymbol{z}$	$\boldsymbol{0}$	$\boldsymbol{c}_N^T - \boldsymbol{y}^T\boldsymbol{N}$	$-\boldsymbol{y}^T\boldsymbol{b}$

- Then, if $\boldsymbol{a}_B^T\boldsymbol{B}^{-1}\boldsymbol{b} + b_{n+1} > 0$, optimality is maintained. If not, we choose this row as the pivot row and apply dual Simplex steps to recover optimality.

Ranging the RHS Parameters

- Ranges for permissible changes to the rhs parameters that maintain feasibility of the optimum solution are computed as follows: From the final tableau, $\tilde{\boldsymbol{b}} = \boldsymbol{S}\boldsymbol{b}$. Assume that the rhs is changed to $\boldsymbol{b} + \Delta$, where $\Delta^{\mathrm{T}} = [\delta_1, \delta_2, \ldots, \delta_m]$. Then, the updated basic solution is given as: $\boldsymbol{S}(\boldsymbol{b} + \Delta)$, where, for feasibility, $\boldsymbol{S}(\boldsymbol{b} + \Delta) \geq 0$ is desired. By inspecting the values in the new $\boldsymbol{x}_{\boldsymbol{B}}$, we can compute the allowable parameter ranges Δ that maintains feasibility.

Ranging the Cost Function Coefficients

- Ranges for permissible changes to the cost function coefficients that maintain feasibility of the optimum solution are computed as follows: Assume that $\boldsymbol{c}^T$ is changed to $\boldsymbol{c}^T + \Delta$, where $\Delta^{\mathrm{T}} = [\delta_1, \delta_2, \dots, \delta_m]$. Then, the updated reduced costs are given as: $\hat{\boldsymbol{c}}^T - \Delta$. This would affect the basis vectors. Gauss-Jordan eliminations are then used to regain the basis. The resulting rhs coefficients are checked for feasibility. By inspecting the values in the new $\boldsymbol{x_B}$, we can compute the allowable parameter ranges Δ that maintains feasibility.

Example

- Consider a standard LP problem:

$$\min_{x_1,x_2} z = -3x_1 - 2x_2$$

Subject to: $2x_1 + x_2 + s_1 = 12,\ \ 2x_1 + 3x_2 + s_2 = 16;\ \ x_j,\ s_j \geq 0$

- The initial and final tableaus for the problem are given as:

Basic	x_1	x_2	s_1	s_2	Rhs
s_1	2	1	1	0	12
s_2	2	3	0	1	16
$-z$	-3	-2	0	0	0

Basic	x_1	x_2	s_1	s_2	Rhs
x_1	1	0	0.75	-0.25	5
x_2	0	1	-0.5	0.5	2
$-z$	0	0	1.25	0.25	19

- Then, $\boldsymbol{S} = \begin{bmatrix} 0.75 & -0.25 \\ -0.5 & 0.5 \end{bmatrix}$, $\boldsymbol{y}^T = [1.25 \quad 0.25,]$, $\boldsymbol{Sb} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$, $\boldsymbol{y}^T\boldsymbol{b} = 19.$
- For a change $\Delta = \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix}$ to RHS, $\boldsymbol{S}(\boldsymbol{b} + \Delta) = \begin{bmatrix} 5 + 0.75\delta_1 - 0.25\delta_2 \\ 2 - 0.5\delta_1 + 0.5\delta_2 \end{bmatrix} \geq 0$
- By inspection, we determine: $-6.67 \leq \delta_1 \leq 4;\ -4 \leq \delta_2 \leq 20$

Example: Postoptimality Analysis

- A vegetable farmer has the choice to grow three vegetables: tomatoes, green peppers, or cucumbers on his 200 acre farm. The man-days/per acre needed for growing the vegetables are 6,7 and 5, respectively. A total of 500 man-hours are available. The yield/ acre for the vegetables are in the ratios: 4.5:3.6:4.0. Determine the optimum crop combination that maximizes total yield.
- The initial and the final tableaus for the problem are given as:

Basic	x_1	x_2	x_3	s_1	s_2	Rhs
s_1	1	1	1	1	0	200
s_2	6	7	5	0	1	500
$-z$	-4.5	-3.6	-4	0	0	0

Basic	x_1	x_2	x_3	s_1	s_2	Rhs
s_1	-0.2	-0.4	0	1	-0.2	100
x_3	1.2	1.4	1	0	0.2	100
$-z$	0.3	2	0	0	0.8	400

Example: Postoptimality Analysis

- From the final tableau, the optimum crop combination is given as: $x_1^* = 0, x_2^* = 0, x_3^* = 100$, with $z^* = 400$. The simplex multipliers for the constraints are: $\boldsymbol{y}^T = [0,\ 0.8]$, with $z^* = \boldsymbol{y}^T\boldsymbol{b} = 400$.
- Using the information in the final tableau, we answer the following:
  - If an additional 50 acres are added, what is the expected change in yield? The answer is found from: $z^* = \boldsymbol{y}^T(\boldsymbol{b} + \Delta)$ for $\Delta = [50,0]^T$, with $z^* = 400$, i.e., there is no expected change in yield. Thus, the land area constraint is not binding in the optimum solution.
  - If an additional 50 man-days are added, what is the expected change in yield? The answer is found from: $z^* = \boldsymbol{y}^T(\boldsymbol{b} + \Delta)$ for $\Delta = [0, 50]^T$, with $z^* = 440$, i.e., the yield increases by 40 units. Thus, the man-days constraint is binding in the optimum solution.

Postoptimality analysis

- If the yield/acre for tomatoes increases by 10%, how is the optimum affected? The answer is found by re-computing the reduced costs as: $\hat{\boldsymbol{c}}^T = \boldsymbol{y}^T\boldsymbol{A} - \boldsymbol{c}^T = [-0.15, 2, 0]$. Since a reduced cost is now negative, additional Simplex steps are needed to regain optimality. This is done and the new optimum is: $x_1^* = 83.33, x_2^* = 0, x_3^* = 0$ with $z^* = 412.5$.
- If the yield/acre for cucumbers drops by 10%, how is the optimum be affected? The answer is found by re-computing the reduced costs as: $\hat{\boldsymbol{c}}^T = \boldsymbol{y}^T\boldsymbol{A} - \boldsymbol{c}^T = [0.3, 2, 0.4]$. The reduced costs are non-negative, but x_3 is no more a basic variable. Regaining the basis results in reduced cost for x_1 becoming negative. Additional Simplex steps to regain optimality reveal the new optimum: $x_1^* = 83.33, x_2^* = 0, x_3^* = 0$ with $z^* = 375$.
- If the yield/acre for green peppers increases to 5/acre, how is the optimum affected? The answer is found by re-computing the reduced cost: $\hat{c}_2 = y^T A_2 - c_2 = 0.4$. Since x_2 was non-basic and the revised reduced cost is non-negative, there is no change in the optimum solution.

Example: Postoptimality Analysis

- Consider the LP problem:

  $$\max_{x_1 - x_4} z = 4x_1 + 6x_2 + 10x_3 + 9x_4$$

  Subject to: $3x_1 + 4x_2 + 8x_3 + 6x_4 \leq 400$, $6x_1 + 2x_2 + 5x_3 + 8x_4 \leq 400$; $x_j \geq 0, j = 1 - 4$

- The final tableau for the problem is given as:

Basic	$\boldsymbol{x_1}$	$\boldsymbol{x_2}$	$\boldsymbol{x_3}$	$\boldsymbol{x_4}$	$\boldsymbol{s_1}$	$\boldsymbol{s_2}$	Rhs
$\boldsymbol{x_1}$	0.75	1	2	1.5	0.25	0	100
$\boldsymbol{s_2}$	4.5	0	1	5	-0.5	1	200
$\boldsymbol{-z}$	0.5	0	2	0	1.5	0	600

- Then, $S = \begin{bmatrix} 0.25 & 0 \\ -0.5 & 1 \end{bmatrix}, \tilde{b} = \begin{bmatrix} 100 \\ 200 \end{bmatrix},$

Example

- (a) How many units of P1, P2, P3 and P4 should be produced in order to maximize profits?
- (b) Assume that 20 units of P3 have been produced by mistake. What is the resulting decrease in profit?
- (c) In what range can the profit margin per unit of P1 vary without changing the optimal basis?
- (d) In what range can the profit margin per unit of P2 vary without changing the optimal basis?
- (e) What is the marginal value of increasing the production capacity of Workshop 1?
- (f) In what range can the capacity of Workshop 1 vary without changing the optimal basis?
- (g) Management is considering the production of a new product P5 that would require 2 hours in Workshop 1 and ten hours in Workshop 2. What is the minimum profit margin needed on this new product to make it worth producing?

Example

- (a) $x_1 = 0,\ \ x_2 = 100, x_3 = 0, x_4 = 0$
- (b) $\delta f = -\hat{c}_3 x_3 = -40$
- (c) For $c_1 = 4 + \delta c_1, \hat{c}_1 = 0.5 - \delta c_1 \geq 0; \delta c_1 \leq 0.5$
- (d) For $c_2 + \delta c_2, \hat{c}_2 = -\delta c_2; y^T = [1.5 + \delta c_2,\ \ 0];\ \hat{\boldsymbol{c}}^T = \boldsymbol{y}^T \boldsymbol{A} - \boldsymbol{c}^T \geq 0$
- (e) $y_1 = 1.5$
- (f) $S\left(\boldsymbol{b} + \begin{bmatrix}\delta b_1 \\ 0\end{bmatrix}\right) = \begin{bmatrix}100 + .25\delta b_1 \\ 200 - .5\delta b_1\end{bmatrix} \geq 0$
- (g) $\boldsymbol{A}_5 = \begin{bmatrix}2 \\ 10\end{bmatrix}; \boldsymbol{y}^T \boldsymbol{A}_5 - c_5 < 0$

Duality in LP Problems

- Associated with every LP problem, termed primal, is a dual problem modeled with dual variables
- The dual problem has one variable for each constraint in the primal and one constraint for each variable in the primal.
- The dual variables are the cost of resources in the primal problem.
- If the primal problem is to maximize objective with LE constraints, the dual problem is to minimize dual objective with GE constraints.
- The optimal solutions for the primal and dual problems are equal.
- The computational difficulty of an LP problem is approximately proportional to m^2n; therefore, dual problem may be computationally easier to solve.

Duality in LP Problems

- The primal and dual LP problems are defined as:

(P) $\max_{\boldsymbol{x}} z = \boldsymbol{c}^T\boldsymbol{x}$, subject to $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$, $\boldsymbol{x} \geq \boldsymbol{0}$

(D) $\min_{\boldsymbol{y}} w = \boldsymbol{y}^T\boldsymbol{b}$, subject to $\boldsymbol{y}^T\boldsymbol{A} \geq \boldsymbol{c}^T$, $\boldsymbol{y} \geq \boldsymbol{0}$

Where $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{y} \in \mathbb{R}^m$ denotes primal and dual variables, respectively.

- In the case of standard LP problem, the dual is given as:

(P) $\min_{\boldsymbol{x}} z = \boldsymbol{c}^T\boldsymbol{x}$, subject to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}$

(D) $\max_{\boldsymbol{y}} w = \boldsymbol{y}^T\boldsymbol{b}$, subject to $\boldsymbol{y}^T\boldsymbol{A} \leq \boldsymbol{c}^T$

Where at optimum point: $w_{opt} = \boldsymbol{y}^T\boldsymbol{b} = \boldsymbol{y}^T\mathbf{A}\boldsymbol{x} = \boldsymbol{c}^T\boldsymbol{x} = z_{opt}$

Example: Diet Problem

- Diet problem (var: qty of eggs, cereals, and bread)

  Min = 100*A + 100*B + 50*C;

  3*A + 4*B + 5*C >= 10; (calories)

  5*A + 4*B + 2*C >=10; (protein)

  A + B+ .5*C <= 2.5; (budget)

- Dual problem (var: cost of calories, cost of protein, budget cost)

  Max = 10*X + 10*Y - 2.5*Z;

  3*X + 5*Y - 1*Z <= 100;

  4*X + 4*Y - 1*Z <=100;

  5*X + 2*Y - .5*Z <= 50;

Weak Duality

- Assume that the primal and dual LP problems are defined as:

  (P) $\max_{\boldsymbol{x}} z = \boldsymbol{c}^T\boldsymbol{x}$, subject to $\boldsymbol{Ax} \le \boldsymbol{b}$, $\boldsymbol{x} \ge \boldsymbol{0}$

  (D) $\min_{\boldsymbol{y}} w = \boldsymbol{y}^T\boldsymbol{b}$, subject to $\boldsymbol{y}^T\boldsymbol{A} \ge \boldsymbol{c}^T$, $\boldsymbol{y} \ge \boldsymbol{0}$

- **Weak Duality.** Let $\boldsymbol{x}$ denote a feasible solution to (P) and $\boldsymbol{y}$ denote a feasible solution to (D), then,

  $\boldsymbol{y}^T\boldsymbol{b} \ge \boldsymbol{y}^T\mathbf{A}\boldsymbol{x} \ge \boldsymbol{c}^T\boldsymbol{x}$, i.e., $w(\boldsymbol{y}) \ge z(\boldsymbol{x})$,

- The difference, $\boldsymbol{b}^T\boldsymbol{y} - \boldsymbol{c}^T\boldsymbol{x}$, is referred to as the duality gap.
- Further, if $\boldsymbol{c}^T\boldsymbol{x} = \boldsymbol{b}^T\boldsymbol{y}$, then $\boldsymbol{x}$ is an optimal solution to (P), and $\boldsymbol{y}$ an optimal solution to (D).
- If the primal (dual) problem is unbounded, then the dual (primal) problem is infeasible (i.e., the feasible region is empty)

Strong Duality

- **Strong Duality.** The primal and dual optimum solutions are equal, i.e.,

  $$w_{opt} = \boldsymbol{y}^T \boldsymbol{b} = \boldsymbol{y}^T \mathbf{A}\boldsymbol{x} = \boldsymbol{c}^T \boldsymbol{x} = z_{opt}$$

- Further, if $\boldsymbol{x}$ is the optimal solution to (P), then $\boldsymbol{y}^T = \boldsymbol{c}_B^T \boldsymbol{B}^{-1}$ is the optimal solution to (D), which can be seen from:

  $$w = \boldsymbol{y}^T \boldsymbol{b} = \boldsymbol{c}_B^T \boldsymbol{B}^{-1} \boldsymbol{b} = \boldsymbol{c}_B^T x_{\boldsymbol{B}} = \boldsymbol{c}^T \boldsymbol{x} = z.$$

- The optimality of (P) implies the feasibility of (D), and vice versa. In particular, $\boldsymbol{x} \geq \mathbf{0}$ implies primal feasibility and dual optimality; and, $\hat{\boldsymbol{c}} = \boldsymbol{c} - \boldsymbol{A}^T \boldsymbol{y} \geq \mathbf{0}$ implies primal optimality and dual feasibility.

Complementary Slackness

- At the optimal point, we have: $\boldsymbol{x}^T\boldsymbol{c} = \boldsymbol{x}^T\boldsymbol{A}^T\boldsymbol{y}$, implying:
  $\boldsymbol{x}^T(\boldsymbol{c} - \boldsymbol{A}^T\boldsymbol{y}) = \sum_{\boldsymbol{j}} x_j(c - A^T y)_j = 0$
- Thus, if the jth primal variable is basic, i.e., $x_j > 0$, then the jth dual constraint is binding, i.e., $(A^T y)_j = c_j$; and, if the jth primal variable is non-basic, i.e., $x_j = 0$, then the jth dual constraint is non-binding, i.e., $(A^T y)_j < c_j$.
- Note that primal optimality $(\hat{\boldsymbol{c}} \geq \boldsymbol{0})$ corresponds to dual feasibility $(\boldsymbol{y}^T\boldsymbol{A} \geq \boldsymbol{c}^T)$, and primal feasibility $(\boldsymbol{x} \geq \boldsymbol{0})$ corresponds to dual optimality.

Dual Simplex Method

- The dual simplex algorithm iterates outsides of the feasible region: it initializes with and moves through the dual feasible (primal infeasible) solutions. Thus, the dual simplex method provides a convenient alternative to the two-phase simplex method in the event the optimization problem has no obvious feasible solution
  - The points generated during dual simplex iterations are primal infeasible as some basic variables have negative values.
  - The solutions are always optimal in the sense that the reduced cost coefficients for nonbasic variables are non-negative.
  - An optimal is reached when a feasible solution with non-negative values for the basic variables has been found.

Dual Simplex Algorithm

- Subtract the surplus variables from GE constraints to convert them to equalities; then, multiply those constraints by -1.
- Enter the constraints and the cost function coefficients in a tableau, noting that the initial basic solution is infeasible.
- At each iteration, the pivot element in the dual simplex method is determined as follows:
  - A pivot row A_q^T is selected as the row that has the basic variable with most negative value.
  - The ratio test to select the pivot column is conducted as: $\min_i \left\{ \frac{c_j}{-A_{q,j}} : c_j > 0, A_{q,j} < 0 \right\}$.
- The dual simplex algorithm terminates when the rhs has become non-negative.

Example: dual simplex method

- Consider the primal and dual problems:
- (P) $\max\limits_{x_1,x_2} z = 3x_1 + 2x_2$

  Subject to: $2x_1 + x_2 \leq 12,\ \ 2x_1 + 3x_2 \leq 16;\ \ x_1 \geq 0, x_2 \geq 0$
- (D) $\min\limits_{y_1,y_2} w = 12y_1 + 16y_2$

  Subject to: $2y_1 + 2y_2 \geq 3,\ \ y_1 + 3y_2 \geq 2;\ \ y_1 \geq 0, y_2 \geq 0$
- We subtract surplus variables from the GE constraints and multiply them with -1 before entering them in the initial tableau.

Example: dual simplex method

Basic	y_1	y_2	s_1	s_2	Rhs
s_1	-2	-2	1	0	-3
s_2	-1	-3	0	1	-2
$-w$	12	16	0	0	0

EBV: y_1, LBV: s_1, pivot: (1,1)

Basic	y_1	y_2	s_1	s_2	rhs
y_1	1	1	-1/2	0	3/2
s_2	0	-2	-1/2	1	-1/2
$-w$	0	4	6	0	-18

LBV: s_2, EBV: y_2, pivot: (2,2)

Basic	y_1	y_2	s_1	s_2	Rhs
y_1	1	0	-3/4	1/2	5/4
y_2	0	1	1/4	-1/2	¼
$-w$	0	0	5	2	-19

$y_1^* = 1.25, y_2^* = 0.25, w_{opt} = 19$

Example: dual simplex method

- Note the following in the final tableau:
  - The optimal value of the objective function for (D) is the same as the optimal value for (P).
  - The optimal values for the basic variables for (P) appear as reduced costs associated with non-basic variables in (D)

Non Simplex Methods for LP Problems

- The non-simplex methods to solve LP problems include the **interior-point methods** that iterate through the interior of the feasible region, and attempt to decrease the duality gap between the primal and dual feasible solutions.
- These methods have good theoretical efficiency and practical performance that is comparable with the simplex methods.

Optimality (KKT) Conditions for LP Problems

Consider the LP problem:

$$\max_{\boldsymbol{x}}\ z = \boldsymbol{c}^T\boldsymbol{x}, \quad \text{subject to } \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b},\ \boldsymbol{x} \ge \boldsymbol{0}$$

The Lagrangian function is formed as:

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{v}) = -\boldsymbol{c}^T\boldsymbol{x} - \boldsymbol{u}^T\boldsymbol{x} + \boldsymbol{v}^T(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b} + \boldsymbol{s})$$

The KKT conditions are:

- Feasibility: $\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b} + \boldsymbol{s} = \boldsymbol{0}$
- Optimality: $\nabla_x\mathcal{L}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{A}^T\boldsymbol{v} - \boldsymbol{c} - \boldsymbol{u} = \boldsymbol{0}$
- Complementarity: $\boldsymbol{u}^T\boldsymbol{x} + \boldsymbol{v}^T\boldsymbol{s} = \boldsymbol{0}$
- Non-negativity: $\boldsymbol{x} \ge \boldsymbol{0},\ \boldsymbol{s} \ge \boldsymbol{0},\ \boldsymbol{u} \ge \boldsymbol{0},\ \boldsymbol{v} \ge \boldsymbol{0}$

Dual LP Problem

- Using optimality conditions, the Lagrangian function is written as:

$$\mathcal{L}(\boldsymbol{x}^*, \boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{v}^T(-\boldsymbol{b}) = \Phi(\boldsymbol{v})$$

- Define the dual LP problem:

$$\max_{\boldsymbol{v} \geq \boldsymbol{0}} \Phi(\boldsymbol{v}) = -\boldsymbol{b}^T \boldsymbol{v}$$

Duality Theorem

- The following are equivalent:
  - $\boldsymbol{x}^*$ together with $(\boldsymbol{u}^*, \boldsymbol{v}^*)$ solves the primal problem.
  - The Lagrangian function $\mathcal{L}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{v})$ has a saddle point at $(\boldsymbol{x}^*, \boldsymbol{u}^*, \boldsymbol{v}^*)$, i.e., $\mathcal{L}(\boldsymbol{x}^*, \boldsymbol{u}, \boldsymbol{v}) \leq \mathcal{L}(\boldsymbol{x}^*, \boldsymbol{u}^*, \boldsymbol{v}^*) \leq \mathcal{L}(\boldsymbol{x}, \boldsymbol{u}^*, \boldsymbol{v}^*)$
  - $(\boldsymbol{x}^*, \boldsymbol{u}^*, \boldsymbol{v}^*)$ solves the dual problem: $\max_{\boldsymbol{u} \geq \boldsymbol{0}, \boldsymbol{v}} \mathcal{L}(\boldsymbol{x}^*, \boldsymbol{u}, \boldsymbol{v})$.

    Further, the two extrema are equal, i.e., $\mathcal{L}(\boldsymbol{x}^*, \boldsymbol{u}^*, \boldsymbol{v}^*) = f(\boldsymbol{x}^*)$

Optimality Conditions for QP Problems

Consider the QP problem:

$$\min q(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{c}^T\boldsymbol{x}, \text{ Subject to: } \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}$$

The Lagrangian function is given as:

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{v}) = \frac{1}{2}\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{c}^T\boldsymbol{x} - \boldsymbol{u}^T\boldsymbol{x} - \boldsymbol{v}^T(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b} - \boldsymbol{s})$$

The corresponding KKT conditions are:

- Feasibility: $\boldsymbol{A}\boldsymbol{x} - \boldsymbol{s} = \boldsymbol{b}$
- Optimality: $\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{c} - \boldsymbol{u} - \boldsymbol{A}^T\boldsymbol{v} = \boldsymbol{0}$
- Complementarity: $\boldsymbol{u}^T\boldsymbol{x} + \boldsymbol{v}^T\boldsymbol{s} = \boldsymbol{0}$
- Non-negativity: $\boldsymbol{x} \geq \boldsymbol{0},\ \boldsymbol{s} \geq \boldsymbol{0},\ \boldsymbol{u} \geq \boldsymbol{0},\ \boldsymbol{v} \geq \boldsymbol{0}$

Dual QP Problem

- The dual function for the QP problem is given as:

$$\Phi(\boldsymbol{v}) = \min_{\boldsymbol{x}} \mathcal{L}(\boldsymbol{x}, \boldsymbol{v}) = -\frac{1}{2}\left(\boldsymbol{A}^T\boldsymbol{v} + \boldsymbol{c}\right)^T \boldsymbol{Q}^{-1}\left(\boldsymbol{A}^T\boldsymbol{v} + \boldsymbol{c}\right) - \boldsymbol{v}^T\boldsymbol{b}$$

- In terms of dual function, the dual QP problem is defined as:

$$\max_{\boldsymbol{v} \geq \boldsymbol{0}} \Phi(\boldsymbol{v}) = -\frac{1}{2}\left(\boldsymbol{A}^T\boldsymbol{v} + \boldsymbol{c}\right)^T \boldsymbol{Q}^{-1}\left(\boldsymbol{A}^T\boldsymbol{v} + \boldsymbol{c}\right) - \boldsymbol{v}^T\boldsymbol{b}$$

- The solution to the dual problem is given as:

$$\boldsymbol{v} = -\left(\boldsymbol{A}\boldsymbol{Q}^{-1}\boldsymbol{A}^T\right)^{-1}\left(\boldsymbol{A}^T\boldsymbol{Q}^{-1}\boldsymbol{c} + \boldsymbol{b}\right)$$

$$\boldsymbol{x} = \boldsymbol{Q}^{-1}\boldsymbol{A}^T\left(\boldsymbol{A}\boldsymbol{Q}^{-1}\boldsymbol{A}^T\right)^{-1}\left(\boldsymbol{A}^T\boldsymbol{Q}^{-1}\boldsymbol{c} + \boldsymbol{b}\right) - \boldsymbol{Q}^{-1}\boldsymbol{c}$$

Example: Finite Element Analysis

- Consider the FEA problem: $\min_{\boldsymbol{q}} \prod = \frac{1}{2}\boldsymbol{q}^T\boldsymbol{K}\boldsymbol{q} - \boldsymbol{q}^T\boldsymbol{f}$
- Assume that $\boldsymbol{q}^T = [q_1, q_2]$, and $q_2 \leq 1.2$ is desired
- The stiffness matrix is given as: $\boldsymbol{K} = \frac{10^5}{3}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\frac{N}{m}$
- Let, $\boldsymbol{Q} = \boldsymbol{K},\ \boldsymbol{f} = [P,\ 0]^T,\ \boldsymbol{c} = -\boldsymbol{f},\ \ \boldsymbol{A} = [0 \quad 1],\ \boldsymbol{b} = 1.2;$ then, $\boldsymbol{A}\boldsymbol{Q}^{-1}\boldsymbol{A}^T = 6 \times 10^{-5},\ \boldsymbol{c}^T\boldsymbol{Q}^{-1}\boldsymbol{A}^T = -1.8,\ \boldsymbol{c}^T\boldsymbol{Q}^{-1}\boldsymbol{c} = 1.08 \times 10^{-5}$
- The dual QP problem is defined as:

  $$\max_{\boldsymbol{v} \geq \boldsymbol{0}} \Phi(\boldsymbol{v}) = -3 \times 10^{-5}v^2 - 0.6v - 1.08 \times 10^{-5}$$

- The optimum solution is given as: $v = 1 \times 10^4$; then $q_1 = 1.5\ mm$, $q_2 = 1.2\ mm,\ \prod = 129\ Nm$