Student Research Paper Conference Vol-2, No-24, July 2015

Trajectory Optimization to send a mission in Halo Orbit about L2 Libration Point in Earth-Moon System

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Abstract*— Designing space missions to remain in the vicinity of a Libration point in a three-body system is both useful and more difficult than that for a two-body system. Trajectory and control design history about the unstable Earth-Moon L2 point will become increasingly complex as additional mechanical and scheduling constraints accompany scientific observatory missions. Satisfying such constraints in designing a station-keeping plan may be viewed as an optimization problem, with the objective of maximizing the mission goals. It then adds further complexity to minimize fuel usage as part of the objective, which is always a goal in space mission planning. Solving this design problem is an illustration of the vastness and simplicity of this alternative multiple body mission design approach, which is based on optimal control problem of the whole trajectory and control design.

***Keywords*— Three-body problem, Libration Point, Trajectory Designing, Optimization**

I. INTRODUCTION

In classical Keplarian orbital mechanics we are dealing with two-body problems i.e. the primary or major body and a secondary or orbiting body e.g. satellite. If we take the gravitational force of two bodies apart from our satellite than this becomes a three-body problem. If we make the assumption that the movement of the bodies is strictly circular and that there is no other gravity apart from the two bodies act on the third one then it is called as “Spatial & Circular Restricted Three-Body Problem (CR3BP)”. The equilibrium solution of this CR3BP is the Libration Point also known as Lagrangian Points or Equilibrium point. Libration Points are points in the physical space where two gravitational attractions balance out. For every restricted three-body problem there are five equilibrium points where it is theoretically possible to place spacecraft with tasks that a classical Kepler orbits cannot provide.

Farquhar first studied trajectory design strategies in the locality of Libration points, taking into account periodic instabilities, eccentricity correction, gravitational perturbations and Solar Radiation Pressure (SRP) [1].

Several other methods to compute trajectory control strategies have been developed, for example the Target Point Method [2], and the Floquet Mode Method [3], which incorporates the concept of invariant manifold tubes from dynamical system theory. Recent Genesis Mission also took advantage of these tubes related with Libration point orbits for Mission design [4]. These methods target a trajectory control strategy based on a reasonable estimate from the theory of an orbit that remains about the Libration point only in a simple model, and may optimize maneuver individually from this technique. These strategies do not optimize the design because they do not search the whole acceptable design space. These mission do not require that the spacecraft remain in some particular well-defined orbit that exists only with a simple theoretical model, but the new design methods do not consider acceptable paths other than the reference orbits. It is very likely that this outside space of alternative paths contain a path with a lower associated fuel expenditure, considering a more accurate modeling of the forces and the errors. Thus, the strategy presented in this paper will necessarily results in mission designs with lower fuel costs than mission based on reference orbits strategy.

Proposals of highly constrained missions demands the ability to create mission design strategies that not only keep the spacecraft in the Libration point’s locality, but also optimize for minimum fuel usage and other mission goals. The design method presented in this paper introduces an alternate course in the development of more sophisticated processes for space mission plan. The track is that of concurrent engineering to incorporate optimization from the start of the design process, resulting in lower fuel requirements because of elimination of redundant assumptions about the orbit or the control design (fuel-burning maneuver to keep scheduled orbit). This optimization problem would be renowned from the controls point of view as an ‘Optimal Control Problem’, which simply means we are looking for a control vector that when applied to a dynamical system results in optimum performance as modeled by Objective function [5]. This strategy, of telling what to do, rather than how to do it, has been successfully applied for the design and control of a variety of Earth-orbit in formations [6, 7] and Libration point formations [8]. This paper looks not at formations, but the approach is applied to a single spacecraft system design space with no restrictions on the orbit shape.

The mission design problem studied in this way is a single optimal control problem; a dynamic system that is affected by some selected ;controls’ is solved for its behavior over a time period to find the set of controls that minimize a specific measure if the ‘cost’ of the behavior [9]. The controls in this case are the accerlations added to spacecraft from burning the engines, where the magnitude and the direction over time are the control variables; that are free to be chosen.

Optimal control problems can rarely be solved analytically, which implies that we need to use numerical methods. The first step is to discretize the problem, which is to define the system at discrete points rather than with continuous functions. This results in a finite number of variables. The number of variables for optimal control problem is then the number of variables in the system times the number of discretization points. The consequences of discretizing the optimal control problem explored here is a nonlinearly constraint optimization problem. The unstable dynamics of the L2 vicinity require a more accurate representation of the trajectory to solve the problem than two-body mission design problem. Because the paths are not simple to define mathematically and similar paths can diverge enormously over the long timescales of these mission, the grid representing the position, velocity (state variables) and added accerlation (control variables) at a finite number of time values in the discretized problem must be very fine or very well-designed to enable a numerical solution.

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II. Three-Body Problem

The Circular Restricted Three Body Problem (CR3BP) is defined as a system of two bodies in a circular orbit about their barycenter, and a third body of negligible mass. The equation of motion are solved for the third body. The stationary solution of this problem are the five Libration Points.

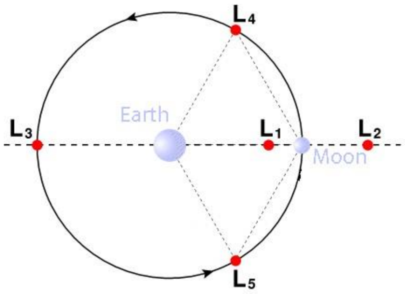

Fig. 1. Libration Points

The equations are most expressed in the rotating barycentric frame (as in this paper). In this frame, the barycenter of the two masses is the origin. The x-axis is the line through the center of two large bodies. The frame rotates about the z-axis with the angular velocity ω of the two large bodies about each other. This angular velocity is then 2π radians divided by the time period of the two-mass system.

With the goal of expressing the accerlation of the third body, we start by taking the derivative with respect to time (in the inertial frame) of the position vector of the third body,

$$\vec{r}\ddot{\vec{r}}_{/I} = \ddot{\vec{r}} + \dot{\vec{\omega}} \times r + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + 2\vec{\omega} \times \dot{\vec{r}} + \ddot{\vec{r}}_B \quad (1)$$

The vector $\vec{r}$ is the position vector with respect to the body frame, whose origin is at the center of the third body in the three-body system. Since the point whose motion we wish to know is the center of the third body, the position vector has zero length since it points from the origin of the body frame to the point in question (i.e. the same point). In evaluating this equation, we note then that $\ddot{\vec{r}}_B = 0$. We also have $\dot{\vec{\omega}} = 0$ because of the circular orbit assumption. It is important to be clear that the derivative of $\vec{r}$ is defined in that frame. Eventually, we want an expression for the accerlation of the position vector with respect to the rotating barycentric frame, so that we can describe the third body’s motion in this frame. Since the angular velocity is about the z-axis, the cross product with ω have simple expressions as seen in the following simplification of the above equation:

$$\ddot{\vec{r}}_{/I} = \ddot{\vec{r}} - \vec{\omega}^2(x\hat{\imath} + y\hat{\jmath}) + 2\vec{\omega}(\dot{y}\hat{\imath} - \dot{x}\hat{\jmath}) \quad (2)$$

We know the acceleration of the position vector in the inertial frame is $\ddot{\vec{r}}_{/I}$ because it is the accerlation on any point in the system due to the gravity of the two large bodies. The force potential at a certain point due to gravity of body 1 is μ_1/r_{1_p} in the direction of body 1, here $\mu_1 = Gm_1$, and r_{1_p} is the distance between body 1 and the point ‘P’; G is the universal gravitational constant, and m_1 is the mass of body 1. We can express $\ddot{\vec{r}}_{/I}$ as the gradient of the total gravitational force potential. First define $\vec{r}_{13}$ as the vector from body 1 to body 3 (body whose motion we are deriving), with the plain $\vec{r}_{13}$ as the distance between the bodies. The same holds for body 2. Now we can write

$$\ddot{\vec{r}}_{/I} = \nabla\left(-\frac{\mu_1}{r_{13}}\vec{r}_{13} - \frac{\mu_2}{r_{23}}\vec{r}_{23}\right) \quad (3)$$

Combining the equation gives

$$\ddot{\vec{r}} - \vec{\omega}^2(x\hat{\imath} + y\hat{\jmath}) + 2\vec{\omega}(\dot{y}\hat{\imath} - \dot{x}\hat{\jmath}) = \nabla\left(-\frac{\mu_1}{r_{13}}\vec{r}_{13} - \frac{\mu_2}{r_{23}}\vec{r}_{23}\right) \quad (4)$$

$\vec{r}_{13}$ as the distance between the bodies. The same holds for body 2. Now we can write

$$\ddot{\vec{r}}_{/I} = \nabla(-\frac{\mu_1}{r_{13}}\vec{r}_{13} - \frac{\mu_2}{r_{23}}\vec{r}_{23}) \quad (5)$$

Combining the equation gives

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Separating this equation in components x,y,z in the rotating frame gives

$$\ddot{x} - \omega^2 x - 2\omega\dot{y} = \frac{\partial}{\partial x}(-\frac{\mu_1}{r_{13}}\vec{r}_{13} - \frac{\mu_2}{r_{23}}\vec{r}_{23}) \quad (7)$$

$$\ddot{y} - \omega^2 y - 2\omega\dot{x} = \frac{\partial}{\partial y}(-\frac{\mu_1}{r_{13}}\vec{r}_{13} - \frac{\mu_2}{r_{23}}\vec{r}_{23}) \quad (8)$$

$$\ddot{z} = \frac{\partial}{\partial z}\left(-\frac{\mu_1}{r_{13}}\vec{r}_{13} - \frac{\mu_2}{r_{23}}\vec{r}_{23}\right) \quad (9)$$

Since

$$r_{13} = \sqrt{(x - r_1)^2 + y^2 + z^2} \quad (10)$$

And

$$r_{23} = \sqrt{(x + r_2)^2 + y^2 + z^2} \quad (11)$$

where, r_j is the distance from the origin to body j, evaluating the differentiation on the right hand sides gives the equation of motions:

$$\ddot{x} - \omega^2 x - 2\omega\dot{y} = -\frac{\mu_1(x-r_1)}{r_{13}^3} - \frac{\mu_2(x-r_2)}{r_{23}^3} \quad (12)$$

$$\ddot{y} - \omega^2 y - 2\omega\dot{x} = -\frac{\mu_1 y}{r_{13}^3} - \frac{\mu_2 y}{r_{23}^3} \quad (13)$$

$$\ddot{z} = -\frac{\mu_1 z}{r_{13}^3} - \frac{\mu_2 z}{r_{23}^3} \quad (14)$$

The non-dimensional units chosen here are TU for time units, DU for distance units, and MU for mass units, $\omega = 1\ \text{radian/TU}$, $m_1 + m_2 = 1\text{MU}$, and $r_{12} = 1\text{D}$. Defining un-subscripted μ as the mass ratio, equals to ${}^{m_2}/_{m_1 + m_2}$ in any units (2 is always the smaller mass), then in non-dimensional units body 1 is also μ DU from the origin and the body 2 is $1 - \mu$ DU from the origin.

In non-dimensional units, r_{13} and r_{23} are

$$r_{13} = \sqrt{(x - r_1)^2 + y^2 + z^2} \quad (15)$$

And

$$r_{23} = \sqrt{(x + r_2)^2 + y^2 + z^2} \quad (16)$$

Evaluating the partial differentials using the above definition gives the non-dimensional equations of motion,

$$\ddot{x} - \omega^2 x - 2\omega\dot{y} = -\frac{(1-\mu)(x-\mu)}{r_{13}^3} - \frac{\mu(x+1-\mu)}{r_{23}^3} \quad (17)$$

$$\ddot{y} - \omega^2 y - 2\omega\dot{x} = -\frac{(1-\mu)y}{r_{13}^3} - \frac{\mu y}{r_{23}^3} \quad (18)$$

$$\ddot{z} = -\frac{(1-\mu)z}{r_{13}^3} - \frac{\mu z}{r_{23}^3} \quad (19)$$

Where, F_i is the component of the sum of the outside forces along the ith axis, and m is the current mass of the spacecraft.

Three different models based on constraints are defined as:

TABLE I. TABLE 1: MISSION MODELS

Simple Model	CR3BP
Perturbed Model	CR3BP + SRP perturbation
Perturbed Mission Constrained Model	CR3BP + perturbations + attitude constrained

III. OPTIMAL CONTROL PROBLEM

The simply constrained set of problems presented here has the following general form: find the state and the control variables at each time step over a given time period, such that a function of the control variables is minimized. The state variables are the position and velocity along the axes, and the mass of the spacecraft, which can be seen as a seven dimensional state vector. To keep the formulation simpler, the attitude is never used as an independent variable, but is constrained in terms of control variables in complex formulation. The control variables are the thrust along the axes, which can be seen as a three-dimensional control vector. Minimizing the function of the control variables then corresponds to minimizing the use of fuel, which allows the launched mass to be minimized as well as opening the possibility for minimizing the time used for station-keeping maneuver that can inhibit payload use. The objective function is the norm of the control vector integrated over the set time period. This optimization is constrained by the equation of motion, which govern the relationship of the position and velocity. We also impose the bound on the distance from the final position to the Libration point, and the constraint of the rocket equation, which governs the relationship between mass and thrust:

$$\frac{\partial m}{\partial t} = -\frac{T}{I_{sp} \times g} \quad (20)$$

The final position which is distance bound (which necessarily occurs at the final time step) ensures the whole trajectory is bounded to meet the station-keeping goal of remaining in the vicinity of L2. There are few more bounds applied to these problems to achieve realistic solutions: the thrust magnitude has a maximum, the x-position is restricted away from the Earth at each time-step, and the mass has a non-zero minimum to restrict the solution from spacecraft consisting entirely of fuel.

The mathematical form of the basic continuous problem is the following optimal control problem: Determine the state-control function pair, s (t), u (t) over $[t_0, t_f]$ that minimize the cost function,

$$J(s,u) = \int_{t_0}^{t_f} F\big(s(t), u(t)\big)dt, \quad (21)$$

Subject to, equation of motion

$$f\big(s(t), u(t)\big) - \dot{s}(t) = 0 \quad (22)$$

boundary constraints,

$$b(s(t_0), s(t_f)) = 0 \quad (23)$$

path constraints

$$h(s(t), u(t)) \leq 0 \quad (24)$$

To frame the current optimal space mission design problem more specifically, the general optimal control problem can be expanded to the following statement. Find the state vector s(t) with elements

$$s = (x(t), y(t), z(t), v_x(t), v_y(t), v_z(t), m(t)) \quad (25)$$

And u(t), the control vector with elements

$$\boldsymbol{u} = (u_x, u_y, u_z) \quad (26)$$

Which minimize the cost,

$$\boldsymbol{J} = \int_{t_0}^{t_f} \|u(t)\|_1 dt \quad (27)$$

This is subject to equation of motion:

$$\dot{x}(t) - v_x(t) = 0 \quad (28)$$

$$\dot{y}(t) - v_y(t) = 0 \quad (29)$$

$$\dot{z}(t) - v_z(t) = 0 \quad (30)$$

$$\dot{v}_x(t) - 2v_y(t) - x(t) + \frac{(1-\mu)(x(t)-\mu)}{r(t)_{13}^3} + \frac{\mu(x(t)+1-\mu)}{r(t)_{23}^3} - \frac{u_x(t)}{m(t)} = 0 \quad (31)$$

$$\dot{v}_y(t) - 2v_x(t) - y(t) + \frac{(1-\mu)y(t)}{r(t)_{13}^3} + \frac{\mu y(t)}{r(t)_{23}^3} - \frac{u_y(t)}{m(t)} = 0 \quad (32)$$

$$\dot{v}_z(t) + \frac{(1-\mu)z(t)}{r(t)_{13}^3} + \frac{\mu z(t)}{r(t)_{23}^3} - \frac{u_z(t)}{m(t)} = 0 \quad (33)$$

$$\dot{m}(t) + \frac{\|u(t)\|_1}{I_{sp} \times g} = 0 \quad (34)$$

The minimization is further subject to the boundary condition:

$$(x(t_f) - (1 - \mu - r_{2-L2}))^2 + y(t_f)^2 + z(t_f)^2 \leq (\alpha L)^2 \quad (35)$$

Where $0 \leq \alpha \leq 1$ and L is the distance between the Earth and L2.

And the path constraints,

$$x(t) \geq 1 - \mu + Margin, (1-\mu) \text{is the position of Earth} \quad (37)$$

$$-T_{max} \leq u(t) \leq T_{max} \quad (38)$$

$$0.01 \leq m(t) \leq 1 \quad (39)$$

where T_{max} is the chosen maximum thrust accerlation.

The equations of motion are expressed in seven equations, with each one describing the change in one of the state variables. The state vector includes the velocity of each position variable in order for the equation of motion to be only first order differential equations.

The boundary condition constraints the final distance from L2 to within some fraction of the distance to the Earth-Moon system. The mass is normalized to be equal to one at the start of the trajectory, which is at t=0 for this optimization problem. Practically, this point may be after a maneuver to insert it in to Libration area trajectory from a launch or transfer trajectory. The mass is constrained to be a minimum of 10% of the mass at the start time, because this is reserved for the mass of the spacecraft itself, as opposed to expendable fuel. This is the conservative estimate for the spacecraft of the size James Webb Telescope has a fuel to total mass ratio of 6%.

The cost function used here is L^1 norm of the control, rather than the L^2 norm squared, which is often used as the familiar quadratic cost function, because L^1 measures fuel use whereas L^2 does not.

$$||u||_{L^p} = \left(\int \left(\sqrt{u_x^2(t) + u_y^2(t) + u_z^2(t)}\right)^p dt\right)^{1/p} \quad (40)$$

or based on the l^1 norm as

$$||u||_{L^p} = \left(\int \left(|u_x(t)| + |u_y(t)| + |u_z(t)|\right)^p dt\right)^{1/p} \quad (41)$$

Using the l^2 based L^1 norm involves a square root calculation, which cause difficulty because of the singularity when u=0. The l^1 based L^1 norm also looks difficult because its derivative is discontinuous at zero, but this was resolved. The choice for the cost function formulation here then is the l^1 based L^1 norm, so that the optimization problem is seeking to minimize the sum of the thrust magnitudes in each direction. The control variables represent the way in which the spacecraft can control its trajectory by adding velocity in the certain direction to its motion. Velocity is added by burning the engine. This cause the thrust in the direction opposite to that in which the engine is pointed. The control vector then represents the thrust, with the components of the vector splitting the thrust magnitude along the axes of the rotating coordinate frame presented earlier. The actual control vector used in formulations here has six components, a positive and negative measure of the thrust along each of the axes (x, y, z). This way of chosen make the problem a reasonable one to solve with the optimization algorithm. The components are then all positive, with lower bound zero and upper bound T_{max}. The most important consequence of this form is the resolution of the l^1 difficulty: it makes the control variables' derivatives continuous. The lack of continuity would have considerably complicated solving the optimization problem. Because the magnitudes of these variables are to be minimized, and in further steps in the mission design process they will be forced to be zero except during planned maneuver times, we know that these variables over the mission lifetime will mostly be zero. Also, only the negative or positive part of the control in one direction can be non-zero at each point in time. This means that the control variables will be on their lower bound (zero) at most times, reducing the number of degrees of freedom in the problem, which makes the optimization problem easier to solve.

IV. TRAJECTORY DESIGN

We solved the initial dynamics of our CR3BP to give us an initial trajectory. What it does is that it gives an initial Δv to our spacecraft from a certain altitude continues its path with the help of gravitational pull of the two primary bodies. The outcome we obtained using MATLAB is given below:

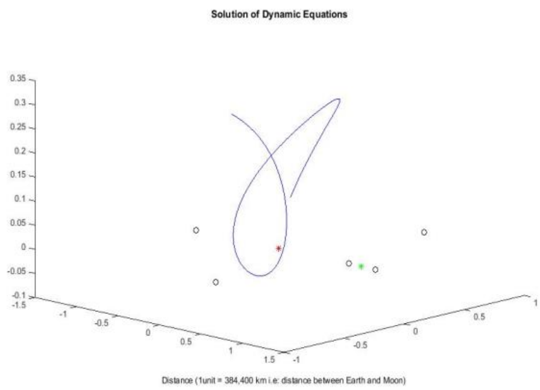

Fig. 2. CR3BP Dynamics with random initial conditions

Now we will solve the given model by giving it some it initial conditions. Above figure was obtained by giving it an arbitrary initial condition. Now we will find the initial conditions so that our trajectory start from GEO orbit and it reaches out final point i.e. the L2 Libration point of Earth-

Moon system with minimum fuel consumption.

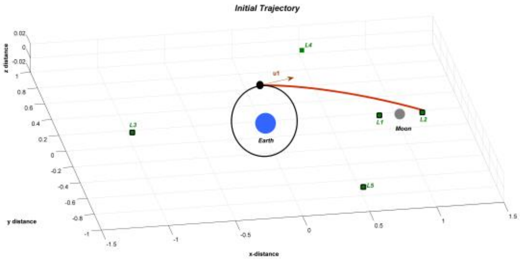

Fig. 3. Initial Trajectory

After having an initial trajectory, our concern now is to minimize the fuel budget of the system i.e. we have to give a minimum Δv for our mission hence, getting an optimal trajectory. For that purpose a MATLAB code was developed by solving it as an optimal control problem and hence, optimizing it to get an optimized trajectory with minimum fuel budget. After solving the optimal control problem the result we obtained is given below.

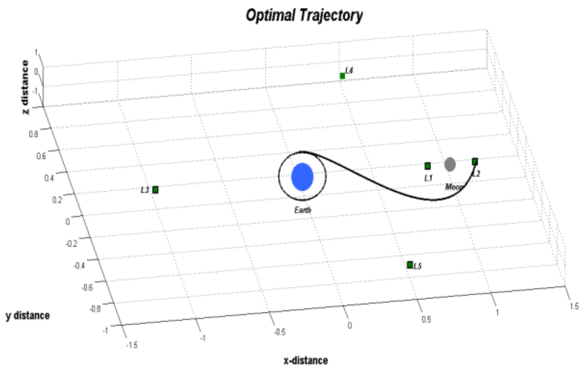

Fig. 4. Optimal Trajectory

V. Conclusion

The results obtained after solving this problem as optimal control problem provide us with an optimal trajectory for which the least amount of fuel will be needed. This result is for a mission from GEO to L2 of Earth-Moon system. It can further be used for a mission to any libration point. Also same method can be applied to any three-body system to get similar results.

In the future a more perturbed model can be used to get more accurate results but for that more computational power will be required. Also a better fuel system with high ISP like ion-propulsion system can be used.

VI. Acknowledgment

We would like to thank ALMIGHTY ALLAH as nothing would have been possible without HIS will and help. We are grateful to our supervisors Dr. Umar Iqbal Bhatti and Dr. Abid Ali Khan for their guidance throughout the course of the project. We are also thankful to Dr Sajjad Asghar and Dr. Abdul Majid for their assistance. We appreciate support of our seniors and class fellows who were available to us throughout the project, guiding us and helping us out where ever we faced any problems.

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