# Mathematical Analysis of Vehicle Lateral Dynamics

[1]Qabool Ahmed, [2]Manohar Lal, [3]Ghulam Abbas, [4]Asad Ahmed, [5]Sunny Deol, [6]Zeeshan Ali
Department Of Mechanical Engineering
Mehran University of Engineering and Technology
Jamshoro, Pakistan
[1]ahmedqabool@gmail.com, [6]zeeshan.ali@faculty.muet.edu.pk

***Abstract*—This paper presents the lateral vehicle dynamic model having two degrees of freedom. The two degrees of freedom are lateral position and yaw angle. The vehicle model has been analyzed for low speed and for a fixed steering angle as a step input. The transient and steady-state response of the vehicle have been investigated on a circular fixed radius track. The simulation results show that vehicle lateral position and yaw angle are linearly increasing under this manoeuver while their derivatives saturate at a fixed values exhibiting the exponential response.**

***Keywords—Lateral dynamics; ; steering angle; bicycle model***

## I. INTRODUCTION

The research paper involves the in depth analysis of lateral motion of the vehicles that is till what extent the effects of lateral dynamics can be eliminated. Lateral motion is in the sideway direction of vehicle that involves two perspectives of motion; one with respect to vehicle itself and other with fixed datum. Undesirable lateral motion of vehicle is mainly due to driver's fatigue, inattention, and drowsiness which are major causes of road accidents. In order to prevent possible accidents due to such motions of vehicle, mathematical model of the lateral motion control system of vehicle is designed and validated in this study.

Our study is based on non-linear model having two degrees of freedom. The dynamical aspect of this motion encounters a great breakthrough as the vehicle is exposed to multiple unbalanced forces that disturbs the orientation of motion. The bad consequences turn up in the form of fatal accidents due to vehicle design errors. It usually occurs if vehicle steers at certain angle or moves up at an inclination. The net output fails to serve yaw control purposed and desired angular motion will leave the vehicle to skid off the path. Yet another significant problem arises when inappropriate values of angular velocity and acceleration severely impact the stability of the vehicle.

Our desired goal is to optimize the dynamical errors caused by the undesired yawing of vehicles. The interpretation of results is obtained by simulations in MATLAB. An authentic solution can be drawn by the simulations to reduce the yawing effects and to foster the stability and controllability. Moreover, a solution is proposed to analytically determine the correct values of velocity at tires for exact steering. The simulation result depicts that the slip angle needs to be reduced to decrease lateral effects and great amount of friction between the tire and road surface is mandatory.

## II. VEHICLE MODEL

The dynamic characteristics of road vehicles are highly dependent on the forces generated between their rolling wheels and the road. In vehicle dynamics we are concerned with the way rolling wheels develop lateral and longitudinal forces and with the effects of these forces on cornering stability and control of road vehicles. When rolling under the condition of small slip, the forces generated by pneumatic tires follow simple mathematical relationships.

When an input to the steering system of an automobile is applied, it results in a change of the direction of travel (yaw angle) and lateral acceleration. At low speed the yaw output is predominant, while at motorway speed, e.g. during a lane change, the lateral acceleration is of primary importance.

In this study a "bicycle" model of the vehicle with two degrees of freedom (DOF) is considered as shown in the Fig. 1. This vehicle model is adopted from [1]. The 2 DOF are represented by the vehicle lateral position $y$ and vehicle yaw angle $\psi$. The vehicle lateral position $y$ is measured along the lateral axis of the vehicle to the point O which is the center of rotation of the vehicle. The vehicle yaw angle $\psi$ is measured with respect to the global $X$ axis. The longitudinal velocity of the vehicle at the C.G. is denoted by $V_x$.

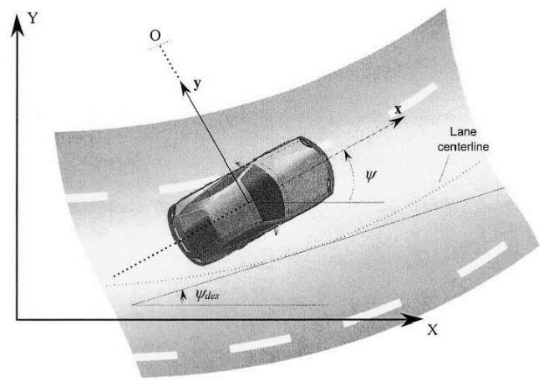


Fig. 1. Vehicle Lateral Dynamics.

Now applying the Newton's second law for motion along the $y$ axis, we get

$$ma_y = F_{yf} + F_{yr} \quad (1)$$

where $a_y$ is the inertial acceleration of the vehicle along the $y$ axis and $F_{yf}$ and $F_{yr}$ are the lateral tire forces at the front and rear wheel respectively. It is important to mention here that two terms contribute to the inertial acceleration; the acceleration $\ddot{y}$ which is due to motion along y axis and the centripetal acceleration $V_x\dot{\psi}$. Therefore, we get

$$a_y = \ddot{y} + V_x\dot{\psi} \quad (2)$$

Substituting (2) in (1), we get the equation for the lateral translational motion of the vehicle as

$$m(\ddot{y} + V_x\dot{\psi}) = F_{yf} + F_{yr} \quad (3)$$

Moreover, the moment balance about the $z$ axis will give the equation for the yaw dynamics as

$$I_z\ddot{\psi} = l_f F_{yf} - l_r F_{yr} \quad (4)$$

where $l_f$ and $l_r$ are the distances of the front tire and rear tire respectively from the C.G. of the vehicle.

In order to find out the solutions of (3) and (4), it is necessary to model the lateral tire forces $F_{yf}$ and $F_{yr}$ that act on the vehicle. In the small range of slip-angles, the experimental results show that these lateral forces of a tire are proportional to the slip-angle [2].

The slip angle of a tire can be defined as the angle between the orientation of the tire and the orientation of the velocity vector of the wheel, See Fig.2. From the Fig. 2, the slip angle of the front wheel is

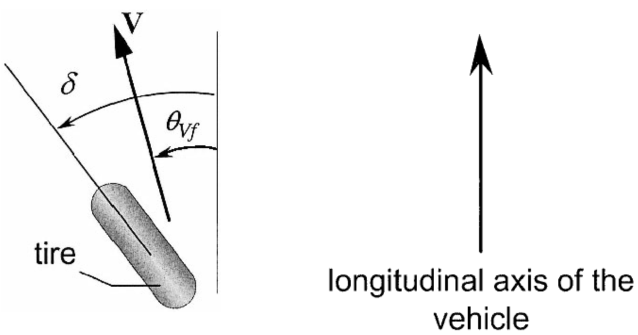


Fig. 2. Tire slip-angle

$$\alpha_f = \delta - \theta_{Vf} \quad (5)$$

where $\theta_{Vf}$ is the angle between the velocity vector with the longitudinal axis of the vehicle and $\delta$ is the front wheel steering angle. Similarly, the rear slip angle is given by

$$\alpha_r = -\theta_{Vr} \quad (6)$$

The lateral tire forces $F_{yf}$ and $F_{yr}$ can now be written as

$$F_{yf} = 2C_{\alpha f}(\delta - \theta_{Vf}) \quad (7)$$

And

$$F_{yr} = 2C_{\alpha r}(-\theta_{Vr}) \quad (8)$$

where the proportionality constants $C_{\alpha f}$ and $C_{\alpha r}$ are the cornering stiffness of each front and rear tire. The factor 2 here accounts for the fact that there are two front and two rear wheels.

Now in order to find out slip angles; $\theta_{Vf}$ *and* $\theta_{Vr}$ the following relations can be used.

$$\tan(\theta_{Vf}) = \frac{V_y + l_f\dot{\psi}}{V_x} \quad (9)$$

$$\tan(\theta_{Vr}) = \frac{V_y - l_r\dot{\psi}}{V_x} \quad (10)$$

Using $V_y = \dot{y}$ and using small angle approximations we get

$$\theta_{Vf} = \frac{\dot{y} + l_f\dot{\psi}}{V_x} \quad (11)$$

$$\theta_{Vr} = \frac{\dot{y} - l_r\dot{\psi}}{V_x} \quad (12)$$

$$\frac{d}{dt}\begin{Bmatrix} y \\ \dot{y} \\ \psi \\ \dot{\psi} \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{-(2C_{\alpha f} + 2C_{\alpha r})}{mV_x} & 0 & \frac{-V_x - (2C_{\alpha f}l_f + 2C_{\alpha r}l_r)}{mV_x} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-(2C_{\alpha f}l_f - 2C_{\alpha r}l_r)}{I_zV_x} & 0 & \frac{-(2C_{\alpha f}l_f^2 + 2C_{\alpha r}l_r^2)}{I_zV_x} \end{bmatrix} + \begin{Bmatrix} 0 \\ \frac{2C_{\alpha f}}{m} \\ 0 \\ \frac{2l_fC_{\alpha f}}{I_z} \end{Bmatrix}\delta \quad (13)$$

Equation (13) describes the state-space model of the vehicle lateral dynamic with two degrees of freedom; vehicle lateral position $y$ and vehicle yaw angle $\psi$ [3]. It is almost not possible to find out the analytical solution of (13), therefore, it can be conveniently solved using numerical method, in this case Runge-Kutta method, with the help of MATLAB programming language.

The numerical integration of (13) yields the values of vehicle's lateral displacement, lateral velocity, yaw angle and yaw velocity.

TABLE I. VEHICLE PARAMETERS

| | | |
|---|---|---|
| Mass of the vehicle | $M$ | 2050 kg |
| Longitudinal speed of the vehicle | $V_x$ | 3 m/s |
| Cornering stiffness of front tire | $C_{af}$ | 77900 N/rad |
| Cornering stiffness of rear tire | $C_{ar}$ | 76500 N/rad |
| Distance from front tire & C.G. of vehicle | $l_f$ | 1.49 m |
| Distance from rear tire & C.G. of vehicle | $l_r$ | 1.71 m |
| Steering angle of the front tire | $\Delta$ | 0.5 radian |
| Moment of inertia of the vehicle | $I_z$ | 5430 kg m$^2$ |

## III. RESULTS AND DISCUSSIONS

This section presents the simulation analysis of bicycle model having two degrees of freedom, i.e. vehicle lateral position $y$ and vehicle yaw angle $\psi$. The initial conditions for both degrees of freedom and their derivatives are zero. The longitudinal speed of the vehicle is 3 m/s (10.8 Km/h), the vehicle has been analyzed for a low speed and under travelling under steady-state conditions. The remaining necessary parameters of the vehicle are given in Table 1.

In this particular scenario, the vehicle is travelling at a constant speed in a circular path of radius 100 m and having a fixed steering angle of 0.5 radian (28.64 degrees) applied as a step input as shown in Fig. 3. The simulation time is 2 s.

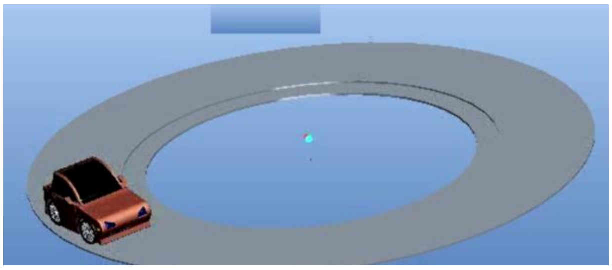

Fig. 3. 3-D CAD Model of the Vehicle

Fig. 4 shows the lateral position of the vehicle which is increasing linearly under the influence of fixed steering angle. It should be noted that the proposed vehicle model is analyzed for a very low speed where the yaw output is more dominant. Fig. 5 shows the transient as well as steady-state response of the lateral velocity of the vehicle against the applied steering angle of 0.5 radians. It can be observed from Fig. 5 that the vehicle response is exponential where due to step input of the steering angle the transient response of the vehicle appears for a very short time. The yaw output is simulated in Fig. 6 and Fig. 7. The orientation of the vehicle is keep changing linearly for the applied steering angle and its derivative saturates at some fixed value. These results help in understanding the response of the lateral dynamics of a passenger car under fixed steer angle which is quite useful for undergraduate students.

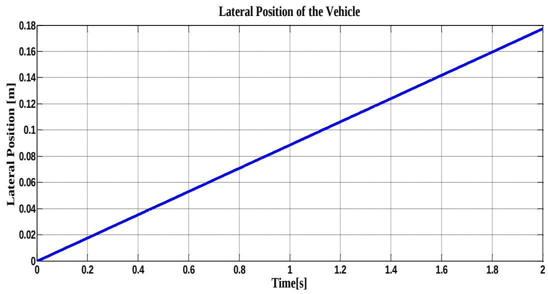


Fig. 4. Lateral Position of the Vehicle

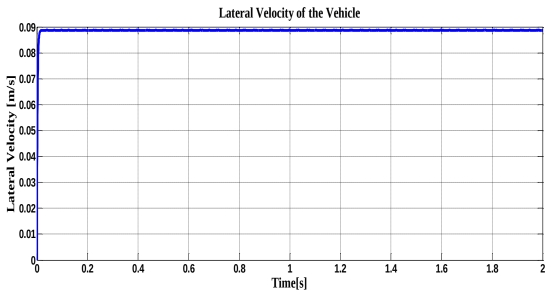


Fig. 5. Lateral Velocity of the Vehicle

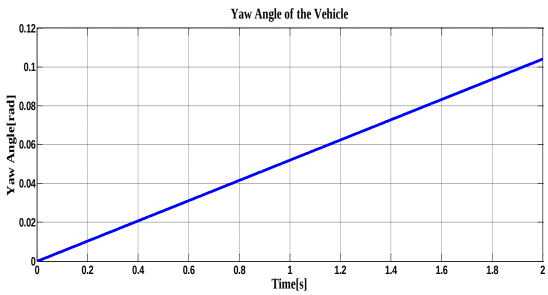


Fig. 6. Yaw Angle of the Vehicle

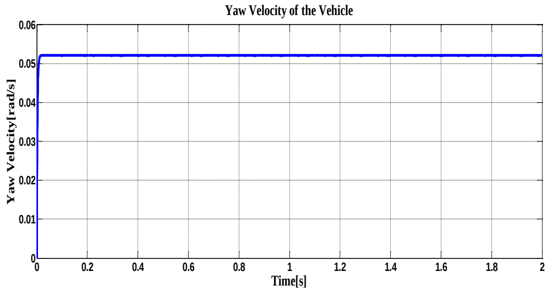


Fig. 7. Yaw Velocity of the Vehicle

## IV. CONCLUSION

The adopted 2 DOF model of the lateral dynamics of the vehicle is presented and simulated in this study. The two 2 DOF are the vehicle lateral position and yaw angle. The vehicle has been analyzed while traveling in a circular path for the applied fixed steering angle. The speed of the vehicle is kept very low to observe the effects of lateral output and yaw output on the vehicle response. For the given parameters of the vehicle given in Table 1, the vehicle response is quite satisfactory for this preliminary investigation. The proposed study is adequately useful for the basic understanding of the lateral dynamics of a vehicle at undergraduate level and can lay down a concrete foundation for the further advancement in the proposed research area.

## V. FUTURE WORK

The presented vehicle model has been analyzed on a flat road. This basic 2DOF mathematical model of the vehicle lateral dynamics can be further improved by incorporating the effects of road bank angle on the performance of the vehicle. Moreover, when it is desired to steer the vehicle for automatic lane keeping then it is essential to redefine the proposed model in terms of position and orientation error with respect to the road. In this way, the error between the actual and desired position and orientation can be measured. A suitable control algorithm can then be proposed to minimize the computed error.

## VI. ACKNOWLEDGMENT

The authors would like to express their thanks to Mechanical Engineering Department, Mehran University of Engineering & Technology, Jamshoro, Pakistan, for giving them an opportunity to pursue this study.

## VII. REFERENCES

[1] R. Rajamani, Vehicle Dynamics and Control. Mechanical Engineering Series. New York, Springer, 2006.

[2] A. Ferrara and C. Vecchio, "Collision Avoidance Strategies and Coordinated Control of Passenger Vehicles." Nonlinear Dynamics vol. 49, 2007, pp. 475-492.

[3] C-W Hong, "Dynamic simulation of road vehicle performance under transient accelerating conditions." Proc Instn Mech Engrs, vol. 210, 1996, pp. 11-21