Model for Vibration of Crack Plates for use with Damage Detection Methodologies

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Abstract— In this paper the equation of motion is derived for a given set of boundary conditions governing the nonlinear vibrations of an isotropic plate with an arbitrarily located part-through crack at the centre of the plate, consisting of a continuous line. The equilibrium principle is used to derive the governing equation of motion in order to get a tractable solution to the vibration problem. Principally, the effects of rotary inertia and through-thickness shear stress are neglected. Galerkin's method is applied to reformulate the governing equation of the cracked plates into time dependent modal coordinates. The simplifying assumptions, and their validity, are described as and when they are made during the derivation of the equations. Berger's formulation is used to generate the form for the in-plane forces and make the model differential equation nonlinear. Results are presented in terms of frequency and half crack length, and found extremely good comparison amongst others.

Index Terms—Isotropic plate, crack, equilibrium principle, in-plane forces, Galerkin's method, vibrations.

4 INTRODUCTION

Machines and structural components potentially require continuous monitoring for the detection of cracks and crack growth for ensuring an uninterrupted service in critical installations. Cracks can be present in structures due to various reasons such as fatigue, impact, corrosion and external and environmental factors like temperature, relative humidity, rainfall and the general properties of structures. Complex structures such as aircraft, ships, steel bridges, sea platforms etc., all use metal plates. The presence of a crack does not only cause a local variation in the stiffness, but can affect the mechanical behaviour of the entire structure to a considerable extent. Cracks present in vibrating components can lead to catastrophic failure as reported by Neogy and Ramamurti [1], Ramamurti and Neogy [2], Trendafilova [3,4], and Trendafilova *et al.* [5] etc. For these reasons, there is a need to understand the dynamics of cracked structures. The vibration characteristics of structures can be useful for on-line detection of cracks without actually dismantling the structure. In particular, the natural frequencies and mode shapes of cracked plates can provide insights into the extent of damage. Israr *et al.* [6,7] studied the dynamics of the cracked plate by considering a part-through crack at the centre of the plate under the application of repeatedly applied periodic force at some specified position with different possible boundary conditions, and proposed the solution of the problem. Israr *et al.* pointed out that crack in the plate influenced the natural frequency of the entire plate structure differently.

In this study an approximate generalized form of the governing differential equation is derived for the damage detection in a rectangular plate having a part-through crack at the centre of the plate subjected to harmonic load with three sets of boundary conditions and aspect ratios.

5 AN APPROXIMATE GENERALIZED FORM OF CRACK RECTANGULAR PLATE

5.1 Overview and Developing Equation

The classical form of the governing equation of rectangular plate is rigorously treated by Timoshenko [8], Leissa [9], and Szilard [10], and so, by neglecting the effect of rotary inertia and through-thickness shear forces, it can be written as:

$$D\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right) = -\rho h\frac{\partial^2 w}{\partial t^2}+n_x\frac{\partial^2 w}{\partial x^2} + n_y\frac{\partial^2 w}{\partial y^2}+2n_{xy}\frac{\partial^2 w}{\partial x\partial y}+P_z \tag{1}$$

where w is the transverse deflection, P_z is the load per unit area acting at the surface, ρ is the density, h is the thickness of the plate and n_x, n_y, and n_{xy} are the in-plane or membrane forces per unit length. D is the flexural rigidity and can be defined as $D=Eh^3/12(1-\upsilon^2)$; E is the modulus of elasticity, and υ is the Poisson's ratio.

Initially, the derivation of the governing equation of the plate having a part-through crack consisting of a continuous line of length $2a$, located at the centre and parallel to the x-direction of the plate as depicted in Fig. 1 is performed by considering that the cracked plate is linear with the following basic assumptions [6]. Later, the governing equation of the cracked plate transforms into nonlinear form by the application of Berger's formulation [11].

1. The plate is made of a perfectly elastic, homogeneous, isotropic material and has a uniform thickness h which is considered small in comparison with its other dimensions.
2. All strain components are small enough to allow Hooke's law to hold.
3. The normal stress component in the direction transverse to the plate surface is small compared with other stress

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components, and is neglected in the stress-strain relationship.

4. Shear deformation is neglected in this case and it is assumed that sections taken normal to the middle surface before deformation remain plane and normal to the deflected middle surface of the plate.

5. The effect of the rotary inertia, shear forces and in-plane force in the y-direction i.e. n_y and n_{xy} are neglected mainly to make the problem more tractable.

For relatively thick plates *(h/l>2)*, where h is the plate thickness and l is an average length in its plane, the effects of shear deformation and rotary inertia become significant, as explained by Leissa [12]. Moreover, in vibration problems, the effect of rotary inertia and shear deformation corresponding to higher modes are more pronounced than on those corresponding to lower modes, and also yields mathematical complexity. In this study, the first mode is discussed in more detail, therefore the assumption made that the effect of the rotary inertia, and shear forces are both negligible is applicable in the subsequent derivation.

The equilibrium equations are obtained by resolving the forces in the z-direction and taking moments about the x and y-axes. The forces acting on the plate element are shown in Fig. 1.

Summing the forces along the z-axis leads to,

$$\begin{aligned}\sum F_z = 0; -Q_x\,dy + \left(Q_x + \frac{\partial Q_x}{\partial x}dx\right)dy - Q_y dx \\ + \left(Q_y + \frac{\partial Q_y}{\partial y}dy\right)dx + P_z dxdy \\ = \rho h \frac{\partial^2 w}{\partial t^2} dxdy\end{aligned} \tag{2}$$

where Q_x and Q_y are the forces per unit length which are projected along z direction, ρ is the density, h is the thickness and P_z is the load per unit area acting over the surface of the plate. Later, this P_z is replaced by a point load $\bar{P}_z$ based on the application of the appropriate delta function because, in practice, it is straightforward to implement this type of loading. Therefore,

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = \rho h \frac{\partial^2 w}{\partial t^2} - P_z \tag{3}$$

The moment equilibrium about the y-axis gives,

$$\begin{aligned}\sum M_y = 0; -M_x dy + \left(M_x + \frac{\partial M_x}{\partial x}dx\right)dy - M_{yx}dx \\ + \left(M_{yx} + \frac{\partial M_{yx}}{\partial y}dy\right)dx \\ - \left(Q_x + \frac{\partial Q_x}{\partial x}dx\right)\frac{dx}{2}dy - Q_x dy\frac{dx}{2} \\ = 0\end{aligned} \tag{4}$$

After simplification, small quantity of higher-order is neglected. Therefore,

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{yx}}{\partial y} = Q_x \tag{5}$$

and hence

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_{yx}}{\partial y \partial x} = \frac{\partial Q_x}{\partial x} \tag{6}$$

Similarly, the moment equilibrium about the x-axis can be written as,

$$\begin{aligned}\sum M_x = 0; \left(M_y + \bar{M}_y\right)dx \\ - \left(M_y + \frac{\partial M_y}{\partial y}dy + \bar{M}_y + \frac{\partial \bar{M}_y}{\partial y}dy\right)dx \\ - M_{xy}dy + \left(M_{xy} + \frac{\partial M_{xy}}{\partial x}dx\right)dy \\ + \left(Q_y + \frac{\partial Q_y}{\partial y}dy\right)\frac{dy}{2}dx + Q_y dx\frac{dy}{2} \\ = 0\end{aligned} \tag{7}$$

After simplification, small quantity of higher-order is neglected. Therefore,

$$-\frac{\partial M_y}{\partial y} - \frac{\partial \bar{M}_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} = -Q_y \tag{8}$$

and hence,

$$-\frac{\partial^2 M_y}{\partial y^2} - \frac{\partial^2 \bar{M}_y}{\partial y^2} + \frac{\partial^2 M_{xy}}{\partial x \partial y} = -\frac{\partial Q_y}{\partial y} \tag{9}$$

Now, substituting Eqs. (6) and (9) into Eq. (3), gives,

$$\begin{aligned}\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + \frac{\partial^2 \bar{M}_y}{\partial y^2} \\ = \rho h \frac{\partial^2 w}{\partial t^2} - P_z\end{aligned} \tag{10}$$

where M_x, M_y and M_{xy} are the bending moments per unit length along the x and y directions. $\bar{M}_y$ is the bending moment per unit length due to the crack at the centre of the plate. Expressing the moments in terms of the curvatures leads to the following result,

$$\begin{aligned}D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) \\ = -\rho h \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2 \bar{M}_y}{\partial y^2} + P_z\end{aligned} \tag{11}$$

5.2 Addition of Membrane Forces

Membrane forces occur when the displacements of the plate parallel to its middle surface are constrained by the supports, and assume small displacements throughout. Occasionally, membrane forces apply at the boundaries and are usually caused by temperature variations, pre-stressing and large

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deflection. The magnitude of the membrane forces are a function of the boundary conditions. This is easily visualised by considering two different plates, one clamped along its edges to prevent any translation or rotation and the other simply supported along its edges allowing only rotation. For equal maximum displacements, the deflected surface length of the clamped plate is greater than that of the simply supported plate, resulting in higher membrane forces.

Considering the equilibrium of the dxdy element in Fig. 2, and given that it is subjected to membrane forces n_x, n_y, $n_{xy} = n_{yx}$, and $\bar{n}_y$ (caused by the crack at the centre of the plate) per unit length, then since there are no body forces, the projection of the membrane forces on the x-axis leads to the following form,

$$-n_x dy + \left(n_x + \frac{\partial n_x}{\partial x}dx\right)dy - n_{yx}dx + \left(n_{yx} + \frac{\partial n_{yx}}{\partial y}dy\right)dx = 0 \tag{12}$$

Therefore,

$$\frac{\partial n_x}{\partial x} + \frac{\partial n_{yx}}{\partial y} = 0 \tag{13}$$

Similarly, along the y-axis we find that,

$$\left(-n_y - \bar{n}_y\right)dx + \left(n_y + \frac{\partial n_y}{\partial y}dy + \bar{n}_y + \frac{\partial \bar{n}_y}{\partial y}dy\right)dx - n_{xy}dy + \left(n_{xy} + \frac{\partial n_{xy}}{\partial x}dx\right)dy = 0 \tag{14}$$

leading to,

$$\frac{\partial n_y}{\partial y} + \frac{\partial \bar{n}_y}{\partial y} + \frac{\partial n_{xy}}{\partial x} = 0 \tag{15}$$

The equilibrium of the dxdy element in the z direction is considered next. It is arbitrarily assumed that the left hand and rear edges of the plate element are fixed and lie in the xy plane, as shown in Fig. 3. Other boundary conditions are equally possible. So, after neglecting higher order quantities, we obtain,

$$\sum F_z(x,y) = n_x \frac{\partial^2 w}{\partial x^2} + n_y \frac{\partial^2 w}{\partial y^2} + \bar{n}_y \frac{\partial^2 w}{\partial y^2} + 2n_{xy}\frac{\partial^2 w}{\partial x \partial y} \tag{16}$$

Thus, it can be deduced from Eq. (16) that the effect of the membrane forces on the deflection is equivalent to an assumed lateral force $F_z(x,y)$. Adding this to the lateral forces in the governing equation i.e. Eq. (11), and noting that this force acts only in the x-direction, then the terms in the y and xy directions is neglected; leading to

$$D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) = -\rho h \frac{\partial^2 w}{\partial t^2} + n_x \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 \bar{M}_y}{\partial y^2} + \bar{n}_y \frac{\partial^2 w}{\partial y^2} + P_z \tag{17}$$

This is the equation of motion for the cracked vibrating plate, and for the case of free vibration, $P_z = 0$. The value of w should be such that it must satisfy the boundary conditions at the edges of the plate.

5.3 Formulation of the Crack Terms (($\bar{M}_y$ and $\bar{n}_y$)

Rice and Levy [13] model is based on Kirchoff's bending theory for thin plates and shells and is used here for the formulation of the crack terms as indicated in Eq. (17). They obtained an approximate relationship between nominal tensile and bending stresses at the location of the crack. These relations are taken after some rearrangement, and then by making use of the basic relationships of tensile and bending stresses, it can be deduced that $m_{rs} = 6\,\sigma_{rs}$. A representation of these stresses is given in Fig. 4.

$$\bar{\sigma}_{rs} = \frac{2a}{(6\alpha^{o}_{tb} + \alpha^{o}_{tt})(1-\upsilon^2)h + 2a}\sigma_{rs} \tag{18}$$

and,

$$\bar{m}_{rs} = \frac{2a}{3\left(\frac{\alpha^{o}_{bt}}{6} + \alpha^{o}_{bb}\right)(3+v)(1-v)h + 2a} m_{rs} \tag{19}$$

where $r,s = 1,2$ are intermediate variables required for algebraic simplification. We define $\bar{\sigma}_{rs}$ and $\bar{m}_{rs}$ as the nominal tensile and bending stresses respectively, at the crack location and on the surface of the plate, σ_{rs} and m_{rs} are the nominal tensile and bending stresses at the far sides of the plate, h is the thickness of the plate, a is the half length of the crack, and α^{o}_{bb}, α^{o}_{tt}, $\alpha^{o}_{bt} = \alpha^{o}_{tb}$ are the non-dimensional bending, stretching and stretching-bending compliance coefficients at the centre of the plate element, respectively.

These relationships show that the nominal tensile and bending stresses at the crack location are a function of the nominal tensile and bending stresses at the far side of the plate. It is worth noting that Okamura *et al.* [14] and Khadem and Rezaee [15,16] also restricted their analysis to the effects of bending compliance, and thus avoided the coupling effect by ignoring the stretching compliance. These three compliance coefficients depend upon the crack depth to plate thickness and vanish when crack depth is equal to zero. Rice and Levy [13] model is also shown that in general the compliance coefficient is a function of the ratio of crack depth to plate thickness, and can be calculated at the centre of the cracked plate takes the form, $\alpha^{0}_{rs} = 1.1547\alpha_{rs}$ [6]. The appropriate compliance coefficients, α_{rs}, can then be calculated from the

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relation given in most of the literature such as that of Okamura *et al.* [12], Rice and Levy [11], Khadem and Rezaee [13,14], Israr *et al*. [6,7].

The uniformly distributed tensile and bending stresses as indicated in Eqs. (18) and (19) are at the two sides of the crack location, and these tensile and bending stresses can be expressed in term of tensile and bending force effects. These force and moment is calculated from two-dimensional plane stress plate bending theory, with the cracked section represented as a continuous line spring having its compliance matched to that of the edge cracked strip in plane strain as shown in Fig. 4. Here, it is very useful to mention that the present derivation, and Rice and Levy [13] model, are both based on classical plate theory; therefore the force $\bar{n}_y$ and moment $\bar{M}_y$ in Eq. (17) can be replaced with the new values obtained in Eqs. (18) and (19) with a negative sign, because damage causes a reduction in the overall stiffness of the plate structure, a phenomenon which can also be seen in most of the literature, such as that of Keer and Sve [17], Stahl and Keer [18], Solecki [19], and Khadem and Rezaee [15,16]. Therefore, we can write the tensile and bending stresses as, (Israr *et al.* [6]),

$$\bar{n}_y \equiv -\bar{n}_{rs} = -\frac{2a}{(6\alpha_{tb}^{o}+\alpha_{tt}^{o})(1-v^2)h+2a}\, n_{rs} \tag{20}$$

and,

$$\bar{M}_y \equiv -\bar{M}_{rs} = -\frac{2a}{3\left(\frac{\alpha_{bt}^{o}}{6}+\alpha_{bb}^{o}\right)(3+v)(1-v)h+2a}\, M_{rs} \tag{21}$$

where $\bar{n}_{rs}$ and $\bar{M}_{rs}$ are the force and moment per unit length in the y-direction at the crack location of the plate, respectively, and n_{rs} and M_{rs} are the force and moment per unit length in the y-direction at the far sides of the plate, respectively.

Substituting the values of $\bar{n}_y$ and $\bar{M}_y$ from Eqs. (20) and (21) into Eq. (17), and then plugging in the value of bending stress M_{rs} at the far sides of the plate into Eq. (17), so the governing equation of the plate with a crack at the centre extends to the following form,

$$\begin{aligned} &D\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2 \partial y^2}+\frac{\partial^4 w}{\partial y^4}\right) \\ &= -\rho h\frac{\partial^2 w}{\partial t^2}+n_x\frac{\partial^2 w}{\partial x^2}+P_z \\ &+\frac{2aD\left(\frac{\partial^4 w}{\partial y^4}+v\frac{\partial^4 w}{\partial y^2 \partial x^2}\right)}{3\left(\frac{\alpha_{bt}^{o}}{6}+\alpha_{bb}^{o}\right)(3+v)(1-v)h+2a} \\ &-\frac{2a}{(6\alpha_{tb}^{o}+\alpha_{tt}^{o})(1-v^2)h+2a}\, n_{rs}\frac{\partial^2 w}{\partial y^2} \end{aligned} \tag{22}$$

6 GALERKIN'S METHOD FOR A VIBRATING CRACKED PLATE

Solutions based on linear models are considered adequate for many practical and engineering purposes although it is recognized that linearised equations usually provide no more than a first approximation. Linearised models of vibrating systems are inadequate in cases where displacements are not small. In addition, problems treated by nonlinear theory exhibit new phenomena, for example the dependence of frequency of vibration on amplitude that cannot be predicted by means of linear theories. Moreover, an example of such a source of nonlinearity is a crack within a plate, which can lead to profound changes in the vibrational response of the system.

Galerkin's Method is applied to reformulate the governing equation of the cracked plate (Eq. (22)) into time dependent modal coordinates by the application of given boundary conditions, and Berger's formulation [11] is used to express formally the in-plane forces, which can then be used to transform the governing equation of the cracked plate into a nonlinear system. To accomplish this we consider the rectangular plate of Fig. 5, of length l_1 in the x-direction and l_2 in the y-direction containing a crack which consists of a continuous line of length $2a$ located at the centre and parallel to the x-direction of the plate. A point load $\bar{P}_z$ based on the application of the appropriate delta function is introduced at the arbitrary location of (x_o, y_o).

The solution for the governing differential equation of the plate subjected to transverse loading is obtained by defining the characteristic functions depending upon the boundary conditions of the plate. The basic model for solution is the one in which all edges are simply supported, while for other boundary conditions the principle of superposition holds (Timoshenko [8], and Berthelot [20]). The most general form of the transverse deflection of the plate is,

$$w(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn} X_m Y_n\, \phi_{mn}(t) \tag{23}$$

where X_m and Y_n are the characteristic or modal functions of the cracked rectangular plate, A_{mn} is an arbitrary amplitude and $\phi_{mn}(t)$ is the time dependent modal coordinate.

The appropriate expressions for the characteristic or modal functions are the one that satisfy the boundary conditions of the plate. Three boundary conditions i.e. Clamped-Clamped-Free-Free (CCFF), Clamped-Clamped-Simply Supported-Simply Supported (CCSS), and all sides Simply Supported has been discussed in the proceeding sections, and these boundary conditions have been treated by different researchers such as the case in which two adjacent edges are clamped while the other two edges are free (CCFF) was examined by Thimoshenko [8], Young [21], Nagaraja and Rao [22], in the monograph of Leissa [9], and Berthelot [20]. The condition in which two adjacent edges are clamped while the other two edges are simply supported (CCSS) was examined by Iwato [23], and the one in which all sides are simply supported (SSSS) was studied by Szilard [10], and Yagiz and Sakman [24]. The lateral load $\bar{P}_z$ at position (x_o, y_o) is defined by Fan

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[25] as $\overline{P_z} = P_o(t)\delta(x-x_o)\delta(y-y_o)$. Substituting the definition of $w(x,y,t)$ from Eq. (23) and the value of $\overline{P_z}$ into Eq. (22) and rearranging the terms, we get,

$$D\left(\frac{\partial^4 X_m}{\partial x^4}Y_n + 2\frac{\partial^4 X_m Y_n}{\partial x^2 \partial y^2} + \frac{\partial^4 Y_n}{\partial y^4}X_m - n_x\frac{\partial^2 X_m}{\partial x^2}Y_n + \frac{2a}{(6\alpha^0_{tb+}\alpha^0_{tt})(1-v^2)h+2a}n_{ab}\frac{\partial^2 Y_n}{\partial y^2}X_m - \frac{2a}{3\left(\frac{\alpha^o_{bt}}{6}+\alpha^o_{bb}\right)(3+v)(1-v)h+2a}D\frac{\partial^4 Y_n}{\partial y^4}X_m + v\frac{\partial^4 X_m Y_n}{\partial y^2 \partial x^2}\right)A_{mn}\phi(t) = -\rho h\frac{\partial^2\phi(t)}{\partial t^2}A_{mn}X_mY_n + P_o(t)\delta(x-x_o)\delta(y-y_o) \quad (24)$$

Berger's relationship [11] is used to obtain forms for the membrane forces n_x and n_{rs} per unit length in the x and y direction respectively. Berger showed that this approach works well for combinations of simply supported and clamped boundary conditions. This formulation has also been adopted by Wah [26], and Ramachandran and Reddy [27] for determining the nonlinear vibrations of un-damped rectangular plates. The membrane forces can be written after multiply each term by dxdy, then impose the condition that the displacement components vanishes at the external boundaries and around the crack, and applying the definition of $w(x,y,t)$ from Eq. (23), leading to,

$$n_x = DF_{1\_mn}A^2_{mn}\phi^2_{mn}(t) \quad (25)$$

and,
$$n_{rs} = DF_{2\_mn}A^2_{mn}\phi^2_{mn}(t) \quad (26)$$

where the quantity A_{mn} is a modal peak amplitude function, normalised in this case to unity,

$$F_{1\_mn} = \frac{6}{h^2 I_1 I_2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\int_0^{I_1}\int_0^{I_2}\left[\left(\frac{\partial X_m}{\partial x}\right)^2 Y_n^2 + v\left(\frac{\partial Y_n}{\partial y}\right)^2 X_m^2\right]dxdy \quad (27)$$

and,

$$F_{2\_mn} = \frac{6}{h^2 I_1 I_2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\int_0^{I_1}\int_0^{I_2}\left[\left(\frac{\partial Y_n}{\partial y}\right)^2 X_m^2 + v\left(\frac{\partial X_m}{\partial x}\right)^2 Y_n^2\right]dxdy \quad (28)$$

Substituting the membrane forces n_x and n_{rs} from Eqs. (25) and (26) into Eq. (24), multiplying each part of Eq. (24) by the modal function X_m and Y_n for one of the three boundary conditions mentioned above, and then integrating over the plate area, we find that,

$$M_{mn}\ddot{\phi}_{mn}(t) + K_{mn}\phi_{mn}(t) + G_{mn}\phi^3_{mn}(t) = P_{mn} \quad (29)$$

where

$$M_{mn} = \frac{\rho h}{D}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}A_{mn}\int_0^{I_1}\int_0^{I_2}X_m^2 Y_n^2\,dxdy \quad (30)$$

$$K_{mn} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}A_{mn}\int_0^{I_1}\int_0^{I_2}\Bigg(X_m^{\mathrm{iv}}Y_n + 2X_m^{"}Y_n^{"} + Y_n^{iv}X_m - \frac{2a(vX_m^{"}Y_n^{"} + Y_n^{iv}X_m}{3\left(\frac{\alpha^o_{bt}}{6}+\alpha^o_{bb}\right)(3+v)(1-v)h+2a}\Bigg)X_mY_n\,dxdy \quad (31)$$

$$G_{mn} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}A^3_{mn}\int_0^{I_1}\int_0^{I_2}\left(-F_{1\_mn}X_mX_m^{"}Y_n^2 + \frac{2aF_{2\_mn}X_m^2Y_nY_n^{"}}{(6\alpha^o_{tb}+\alpha^o_{tt})(1-v^2)h+2a}\right)dxdy \quad (32)$$

The force term in Eq. (29) can be expressed as

$$P_{mn} = \frac{P_o(t)}{D}Q_{mn} \quad \text{where} \quad Q_{mn} = X_m(x_0)Y_n(y_0) \quad (33)$$

Eq. (29) is in the form of the well known Duffing equation containing a cubic nonlinear term, and can be re-stated as

$$\ddot{\phi}_{mn}(t) + \omega^2_{mn}\phi_{mn}(t) + \beta_{mn}\phi^3_{mn}(t) = \gamma_{mn}P_o(t) \quad (34)$$

where

$$\omega^2_{mn} = \frac{K_{mn}}{M_{mn}}, \qquad \beta_{mn} = \frac{G_{mn}}{M_{mn}}, \qquad \gamma_{mn} = \frac{Q_{mn}}{D\times M_{mn}} \quad (35)$$

and ω_{mn} is the natural frequency of the cracked rectangular plate. β_{mn} is the nonlinear cubic term and can be either a positive (hard spring) or a negative (soft spring) depending upon the system parameters.

Now if it is assumed that the system is attached to a nonlinear spring under the influence of weak linear viscous damping μ, and let the load be harmonic, such that, $P_o(t) = P\cos\Omega_{mn}t$, then the equation of the rectangular cracked plate becomes,

$$\ddot{\phi}_{mn}(t) + 2\mu\dot{\phi}_{mn}(t) + \omega^2_{mn}\phi_{mn}(t) + \beta_{mn}\phi^3_{mn}(t) = \gamma_{mn}P\cos\Omega_{mn}t \quad (36)$$

This problem is not too hard to nondimensionalise, however, physical units of the parameter are used throughout, because there are no significant scale effects, or data complications which would otherwise require the one of formal nondimensionalisation.

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7 RESULTS

A test plate made of aluminium alloy 5083 is used for the comparison of the natural frequencies of cracked and un-cracked plates. This aluminium alloy contains 5.2% magnesium, 0.1% manganese and 0.1% chromium. In the tempered condition, it is strong and retains good formability due to excellent ductility. It has high resistance to corrosion, and is used for various applications such as shipbuilding, aircrafts, rail cars, vehicle bodies, pressure vessels etc. It has low density and excellent thermal conductivity common to all aluminium alloys and has the material properties such as modulus of elasticity, $E = 7.03 \times 10^{10}$ N/m^2, density, $\rho = 2660$ kg/m^3, Poisson's ratio, $\upsilon = 0.33$, and a damping factor of $\mu = 0.08$, while the geometric properties of the test plate are: length along x-direction, l_1 and length along y-direction, l_2 ranges between (0.5-1) m, half crack length, a ranges between (0-0.01) m, and thickness of the plate, $h = 0.01$ m. A point load, $P = 10$ N is chosen, and is acting upon the surface of the plate at some arbitrary specified point given here by $x_o = 0.375$ m and $y_o = 0.75$ m.

The natural frequencies of cracked plates for three sets of boundary conditions i.e. CCFF, CCSS and SSSS with different aspect ratios have been studied and are illustrated in Fig. 6. It can be seen that the presence of the crack at the centre of the plate significantly affects the natural frequency of the first mode of the plate, in all three cases of boundary conditions. The natural frequency is also varied if the geometry of the plate is changed, in particular its length and thickness, in addition to the effect of the half-crack length. It can also been seen that the decrease in the natural frequency as the increase of the half-crack length for the same parameters. These changes are very small for small half-crack lengths, as one would expect.

The present theory can also be verified with existing linear theories as proposed by different investigators, namely, Stahl and Keer [18], Solecki [19], Qian *et al.* [28] , Krawczuk [29], and Krawczuk *et al.* [30] and [31] for the vibration analysis of cracked plates. Let us consider a square plate of sides 0.1 m x 0.1 m made of steel having a material properties such as, young's modulus, $E = 2.04 \times 10^{11}$ N/m^2, Poisson's ratio, $\upsilon = 0.3$, and mass density, $\rho = 7860$ kg/m^3. The thickness is 0.001 m and the plate is simply supported from all sides.

Table 1 presents a comparison of the ratio of frequencies of cracked and un-cracked plates. It shows that the percentage changes between the linear models and present nonlinear model for the range of $2a/l_1$ ratio (0.1-0.2) is approximately (1-2)%.

8 CONCLUSIONS

This study involves the mathematical modeling of vibration in a plate into which a horizontal crack has been introduced. The methodology is principally analytical and has led to a unique solution for this problem in the form of a Duffing equation in modal space. The Duffing equation has not hitherto been shown to be capable of modeling cracked plates. In this work it is shown that different boundary conditions can be admitted for the plate and that the modal natural frequencies are sensitive to the crack geometry.

It can be concluded that conventional methods used for the reduction in frequency response in the cracked plate element, might in fact lead to an increase in the amplitude of excitation for the first mode. In other words, we can say that a loss of local stability of plates with a small crack is possible under periodic loading. Results show that the frequency of the cracked plate changes for each set of aspect ratio and boundary condition differently. Similarly, there is a (1-2)% changes when compared with the results of other investigators.

Finally, this research provides some basic theory and understanding of how nonlinear plate systems can be made to be more efficient. Engineers and scientists could be encouraged to use this new approach for prior understanding of the behaviour of damaged plates and panels. By obtaining a basic understanding, an ideal and robust system can ultimately be configured, and hence more reliable and efficient industrial systems can be constructed for vibration analysis.

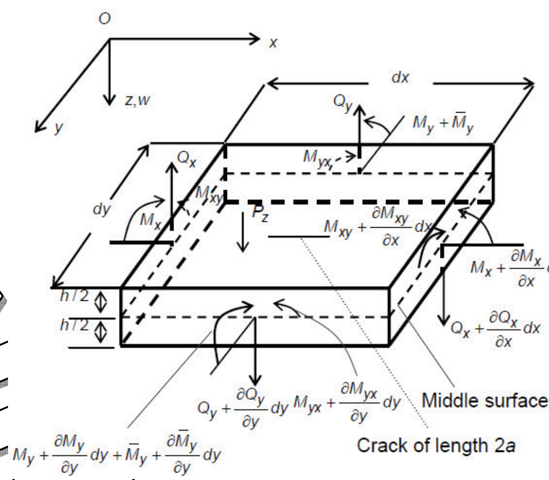

Figure 1 - Isotropic plate loaded by uniform pressure and a small crack at the centre

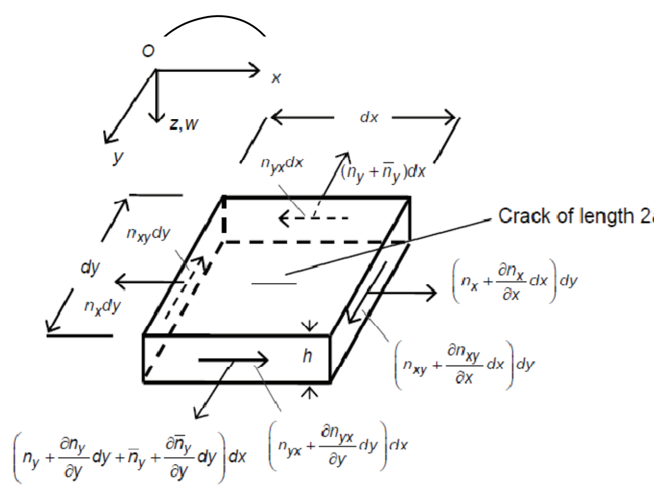

Figure 2 - In-plane forces and a crack of length 2*a* at the centre of the plate element

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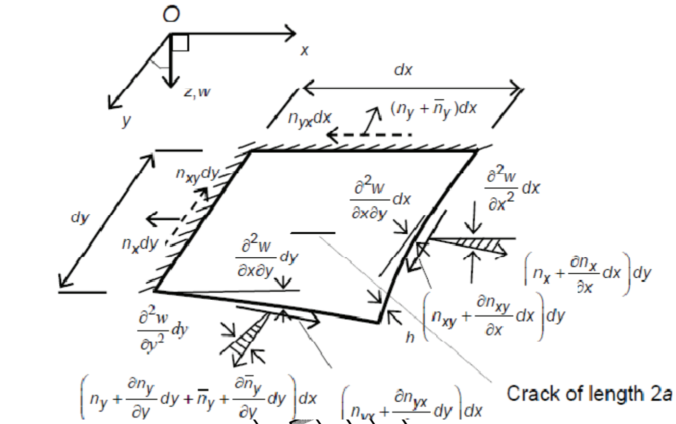

Figure 3 - Two sided constraints and plate deformation having a part-through crack at the centre

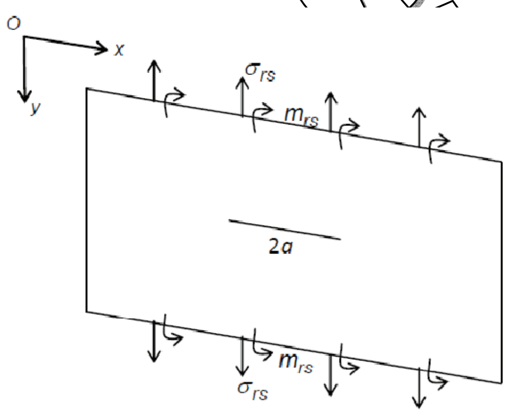

Figure 4 - The bending and tensile stresses for a part-through crack of length 2*a*, after Rice and Levy [13]

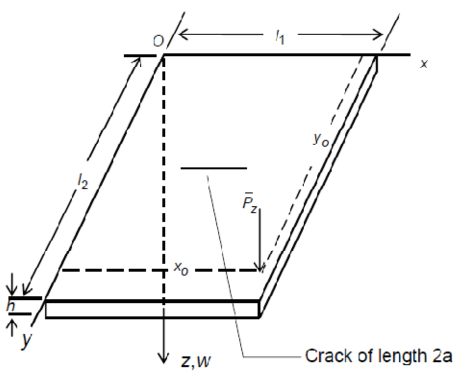

Figure 5 - Isotropic plate having a small crack at the centre under the application of point load at some arbitrary position [6]

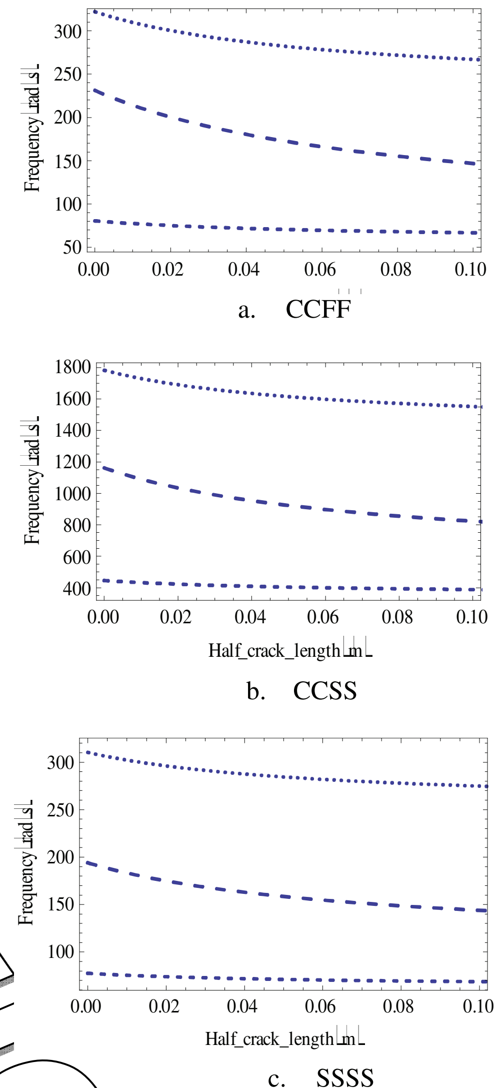

Figure 6 - First mode natural frequencies of cracked plate for three set of boundary conditions (CCFF, CCSS, and SSSS) and aspect ratios.

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Table 1. Relative changes of the natural frequencies of the cracked simply supported plate for the first mode only.

Linear Models	$2a/l_1$				
	0.00	0.05	0.10	0.15	0.20
Stahl and Keer [18]	1	-	0.9940	-	0.9775
Solecki [19]	1	-	0.9940	-	0.9780
Qian *et al.* [28]	1	-	0.9950	-	0.9820
Krawczuk [29]	1	0.9971	0.9942	0.9874	0.9806
Krawczuk *et al.* [30] and [31]	1	0.9971	0.9942	0.9874	0.9806
Present Nonlinear Model	1	0.9995	0.9992	0.9990	0.9989

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